Several Problems in Linear Algebra

Numbers in square brackets next to the task are measuring the complexity (in points out of 5). Some of these questions are comparable with possible theoretical tasks of the final exam, some (mostly those labeled 2+ and higher) are more tricky; let me know if you manage to solve some of them. In the case where the field is not specified explicitly, you may work with either real or complex numbers, whatever you prefer.

1. [1–] Under what conditions the product of two symmetric $n \times n$-matrices A and B is symmetric?

2. [1+] Give an example of two 2×2-matrices having the same rank, trace, and determinant which are not similar to each other.

3. [2–] A, B, C, and D are $n \times n$-matrices. The matrix A is similar to the matrix B, and the matrix C is similar to the matrix D. Is AC necessarily similar to BD?

4. (**a**) [2–] Let A be a 4×4-matrix, and let A_{ij}^{kl} be the matrix that remains after deleting its i^{th} and j^{th} columns, and k^{th} and l^{th} rows. (For example, $A_{34}^{12} = \left(\begin{smallmatrix} a_{31} & a_{41} \\ a_{32} & a_{42} \end{smallmatrix}\right)$.) Prove the Laplace formula

$$\det(A) = \det(A_{12}^{12})\det(A_{34}^{34}) - \det(A_{13}^{12})\det(A_{24}^{34}) + \det(A_{14}^{12})\det(A_{23}^{34}) + \\ + \det(A_{23}^{12})\det(A_{14}^{34}) - \det(A_{24}^{12})\det(A_{13}^{34}) + \det(A_{34}^{12})\det(A_{12}^{34}).$$

(**b**) [2–] Let A be a 2×4-matrix, and let A_{ij} be the matrix that remains after deleting its i^{th} and j^{th} columns. (For example, $A_{12} = \left(\begin{smallmatrix} a_{31} & a_{41} \\ a_{32} & a_{42} \end{smallmatrix}\right)$.) Prove the Plücker relation

$$\det(A_{12})\det(A_{34}) - \det(A_{13})\det(A_{24}) + \det(A_{14})\det(A_{23}) = 0.$$

5. [2–] Let $n > 2$. For a $n \times n$-matrix A with real entries, show that all $n!$ terms

$$(-1)^{\operatorname{inv}(i_1,\ldots,i_n)} a_{1i_1} a_{2i_2} \ldots a_{ni_n}$$

in the expansion of its determinant cannot be positive at the same time.

6. [1+] Let A be a square matrix of rank 1. Show that $\det(I + A) = 1 + \operatorname{tr}(A)$.

7. [2+] Using the standard inner product $(A, B) = \operatorname{tr}(AB^T)$ on the space of all 2×2-matrices, define the length $\|A\|$ by

$$\|A\| = \sqrt{(A, A)}.$$

Prove that

$$\|AB\| \leqslant \|A\| \cdot \|B\|.$$

8. For a (**a**) [1+] 2×2-matrix; (**b**) [2–] 3×3-matrix A with

$$\operatorname{tr}(A) = \operatorname{tr}(A^2) = \operatorname{tr}(A^3) = \ldots = 0,$$

prove that $A^k = 0$ for some k.

9. [1+] Given two subspaces U_1 and U_2 of a finite-dimensional space V, prove that $\dim(U_1 + U_2) = \dim(U_1) + \dim(U_2) - \dim(U_1 \cap U_2)$.

10. [2–] Given three subspaces U_1, U_2, and U_3 of a finite-dimensional space V, is it possible to compute $\dim(U_1 + U_2 + U_3)$ if you are given $\dim(U_1)$, $\dim(U_2)$, $\dim(U_3)$, $\dim(U_1 \cap U_2$, $\dim(U_1 \cap U_3)$, $\dim(U_2 \cap U_3)$, and $\dim(U_1 \cap U_2 \cap U_3)$?

11. [2–] For two square matrices A and B we have $AB - BA = A$. Show that the matrix A is not invertible.

12. [2+] For an invertible matrix A, the sum of entries in every row is equal to S. Show that the sum of entries in every row of A^{-1} is equal to S^{-1}.

13. [2–] Show that every $m \times n$-matrix A with $\mathrm{rk}(A) = k$ can be represented as a product $A = BC$, where B is a $m \times k$-matrix, and C is a $k \times n$-matrix.

14. [2–] For a square matrix A, $\mathrm{rk}(A) = r$. Compute $\mathrm{rk}(\mathrm{adj}(A))$.

15. [2–] Show that for a $m \times n$-matrix A with $a_{ij} = x_i + y_j$ ($x_1, \ldots, x_m$, $y_1, \ldots, y_n$ are some numbers), we have $\mathrm{rk}(A) \leqslant 2$.

16. [2–] Prove that if the matrix A is nilpotent (that is, $A^k = 0$ for some k), then the matrix $I + A$ is invertible.

17. [1+] Compute the determinant of the matrix

$$\begin{pmatrix} \binom{0}{0} & \binom{1}{1} & \binom{2}{2} & \cdots & \binom{n}{n} \\ \binom{1}{0} & \binom{2}{1} & \binom{3}{2} & \cdots & \binom{n+1}{n} \\ \binom{2}{0} & \binom{3}{1} & \binom{4}{2} & \cdots & \binom{n+2}{n} \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ \binom{n}{0} & \binom{n+1}{1} & \binom{n+2}{2} & \cdots & \binom{2n}{n} \end{pmatrix}$$

18. [3+] Denote by $\varphi(n)$ the "totient function": $\varphi(n)$ is the number of integers between 1 and n that are coprime with n. Let A be the $n \times n$-matrix with $a_{ij} = \gcd(i, j)$ (greatest common divisor of i and j). Show that $\det(A) = \varphi(1)\varphi(2)\ldots\varphi(n)$.

19. (**a**) [2+] Show that if for a $n \times n$-matrix A we have $\mathrm{tr}(A) = 0$, then A can be represented as a sum of several *commutators* $[P_i, Q_i] = P_iQ_i - Q_iP_i$ (where P_i, Q_i are some $n \times n$-matrices.

(**b**) [3+] Show that if for a $n \times n$-matrix A we have $\mathrm{tr}(A) = 0$, then $A = [P, Q]$ for some $n \times n$-matrices P and Q.

20. (**a**) [1–] A $(2n + 1) \times (2n + 1)$-matrix A is skew-symmetric (that is, $A^T = -A$). Show that $\det(A) = 0$.

(**b**) [2+] A $(2n) \times (2n)$-matrix A is skew-symmetric (that is, $A^T = -A$). Show that if we add the same number to all entries of A, the determinant of the resulting matrix is equal to $\det(A)$.

21. [2+] For a $n \times n$-matrix A with real entries and the standard inner product on $\mathbb{R}^n$, it is true that $(Ax, y) = 0$ if and only if $(Ay, x) = 0$. Show that this matrix is either symmetric or skew-symmetric.

22. [2–] Let A be a real symmetric positive definite matrix. Prove that there exists a real symmetric matrix B such that $B^2 = A$.

23. [3–] Let A be a complex square matrix with $\det(A) \neq 0$. Prove that there exists a complex matrix B such that $B^2 = A$.

24. [2+] Give an example of a complex square matrix A for which there is no complex matrix B with $B^2 = A$.

25. [2–] Prove that $\det(e^A) = e^{\mathrm{tr}(A)}$ for any square matrix A.

26. [2+] Given a $n \times n$-matrix A with real entries for which

$$|a_{ii}| > |a_{i1} + a_{i2} + \ldots + a_{i,i-1} + a_{i,i+1} + \ldots + a_{in}|$$

for all $i = 1, 2, \ldots, n$, prove that $\det(A) \neq 0$.

27. For two symmetric real $n \times n$-matrices A and B, we say that $A > B$ if $A - B$ is positive definite. Is it true that

(**a**) [1−] if $A > B$ and $C > D$, then $A + C > B + D$?

(**b**) [1−] if $A > 0$, then $A^2 > 0$?

(**c**) [2−] if $A > I$, then $A^2 > I$?

(**d**) [3−] if $A > B > 0$, then $A^2 > B^2$?

28. [3+] For any two matrices A and B, prove that characteristic polynomials of matrices BA and AB coincide.

29. [2+] Let U_1, U_2, $\ldots$, U_p be subspaces of $\mathbb{R}^n$, and assume that $\dim U_i = k$ for all i. Show that if $U_i \cap U_j$ is $(k-1)$-dimensional for all $i \neq j$, then either $\dim(U_1 \cap U_2 \cap \ldots \cap U_p) = k - 1$, or $\dim(U_1 + \ldots + U_k) \leqslant k + 1$.

30. [2+] Let v_1, $\ldots$, v_k be vectors in a Euclidean space V. Show that if $(v_i, v_j) < 0$ for all $i \neq j$, then this system of vectors becomes linearly independent after throwing away any vector v_i from it.

31. For any finite connected graph Γ with vertices v_1, $\ldots$, v_n (without loops and multiple edges) define a $n \times n$ matrix A_Γ as follows. Put $a_{ii} = 2$, and for $i \neq j$ put

$$a_{ij} = \begin{cases} -1 & \text{if } v_i \text{ is connected with } v_j \text{ by an edge,} \\ 0 & \text{otherwise.} \end{cases}$$

Consider the quadratic form q_Γ on $\mathbb{R}^n$ whose matrix relative to the standard basis is A_Γ.

(**a**) [1+] Prove that some vertex of Γ contains at least 4 outgoing edges, then q_Γ is not positive definite.

(**b**) [1+] Prove that if Γ contains a cycle (of any length), then q_Γ is not positive definite.

(**c**) [2+] Prove that if Γ contains at least two vertices with 3 outgoing edges, the q_Γ is not positive definite.

(**d**) [4−] Describe all graphs Γ for which q_Γ is positive definite.