Jordan normal form for nilpotent operators: some examples

In class, we proved that for every *nilpotent* linear operator A on a vector space V (that is, an operator for which $A^k = 0$ for some k) it is possible to choose a basis

$$\begin{gathered} e_1^{(1)}, e_2^{(1)}, e_3^{(1)}, \ldots, e_{n_1}^{(1)}, \\ e_1^{(2)}, e_2^{(2)}, \ldots, e_{n_2}^{(2)}, \\ \ldots \\ e_1^{(l)}, \ldots, e_{n_l}^{(l)} \end{gathered}$$

of V such that for each "thread"

$$e_1^{(p)}, e_2^{(p)}, \ldots, e_{n_p}^{(p)}$$

we have

$$A(e_1^{(p)}) = e_2^{(p)}, A(e_2^{(p)}) = e_3^{(p)}, \ldots, A(e_{n_p}^{(p)}) = 0.$$

Now we shall consider several examples of how to find such "thread bases".

Example 1. $V = \mathbb{R}^2$, $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

In this case, $A^2 = 0$, $\mathrm{rk}(A) = 1$, $\mathrm{rk}(A^k) = 0$ for $k \geqslant 2$, $\dim \mathrm{Ker}(A) = 1$, $\dim \mathrm{Ker}(A^k) = 2$ for $k \geqslant 2$. Moreover, $\mathrm{Ker}(A) = \{\begin{pmatrix} x \\ 0 \end{pmatrix}\}$.

We have a sequence of subspaces $V = \mathrm{Ker}\, A^2 \supset \mathrm{Ker}\, A \supset \{0\}$. The first one relative to the second one is one-dimensional (since $\dim \mathrm{Ker}\, A^2 - \dim \mathrm{Ker}\, A = 1$). Putting $x = 1$ in the formula above, we get the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ which forms a basis of the kernel of A, and after computing the reduced column echelon form and looking for missing leading 1's, we obtain a relative basis consisting of the vector $f = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. This vector gives rise to a thread $f = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, Af = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ of length 2. Since our space is 2-dimensional, this thread forms a basis.

Example 2. $V = \mathbb{R}^3$, $A = \begin{pmatrix} -3 & 1 & -1 \\ -12 & 4 & -4 \\ -3 & 1 & -1 \end{pmatrix}$.

In this case, $A^2 = 0$, $\mathrm{rk}\, A = 1$, $\mathrm{rk}\, A^k = 0$ for $k \geqslant 2$, $\dim \mathrm{Ker}(A) = 2$, $\dim \mathrm{Ker}(A^k) = 3$ for $k \geqslant 2$. Moreover, $\mathrm{Ker}(A) = \{\begin{pmatrix} \frac{s-t}{3} \\ s \\ t \end{pmatrix}\}$.

We have a sequence of subspaces $V = \mathrm{Ker}\, A^2 \supset \mathrm{Ker}\, A \supset \{0\}$. The first one relative to the second one is one-dimensional (since $\dim \mathrm{Ker}\, A^2 - \dim \mathrm{Ker}\, A = 1$). The kernel of A has a basis consisting of the vectors $\begin{pmatrix} 1/3 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1/3 \\ 0 \\ 1 \end{pmatrix}$ (corresponding to the choices $s = 1, t = 0$ and $s = 0, t = 1$ respectively), and after computing the reduced column echelon form and looking for missing leading 1's, we obtain a relative basis consisting of the vector $f = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. This vector gives rise to the thread f, $Af = \begin{pmatrix} -1 \\ -4 \\ -1 \end{pmatrix}$. It remains to find a basis of $\mathrm{Ker}\, A$ relative to the span of Af. Column reduction of the basis vectors of $\mathrm{Ker}(A)$ by Af leaves us with the vector

$g = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$. Overall, f, Af, g form a basis of V. It consists of two threads, one of length 2 (f, Af) and the other one of length 1 (g).

Example 3. $V = \mathbb{R}^3$, $A = \begin{pmatrix} 21 & -7 & 8 \\ 60 & -20 & 23 \\ -3 & 1 & -1 \end{pmatrix}$.

In this case, $A^2 = \begin{pmatrix} -3 & 1 & -1 \\ -9 & 3 & -3 \\ 0 & 0 & 0 \end{pmatrix}$, $A^3 = 0$, $\operatorname{rk} A = 2$, $\operatorname{rk} A^2 = 1$, $\operatorname{rk} A^k = 0$ for $k \geqslant 3$, $\dim \operatorname{Ker}(A) = 1$, $\dim \operatorname{Ker}(A^2) = 2$, $\dim \operatorname{Ker}(A^k) = 3$ for $k \geqslant 3$.

We have a sequence of subspaces $V = \operatorname{Ker} A^3 \supset \operatorname{Ker} A^2 \supset \operatorname{Ker} A \supset \{0\}$. The first one relative to the second one is one-dimensional ($\dim \operatorname{Ker} A^3 - \dim \operatorname{Ker} A^2 = 1$). We have $\operatorname{Ker}(A^2) = \{\begin{pmatrix} \frac{s-t}{3} \\ s \\ t \end{pmatrix}$, so it has a basis of the vectors $\begin{pmatrix} 1/3 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1/3 \\ 0 \\ 1 \end{pmatrix}$ (corresponding to the choices $s = 1, t = 0$ and $s = 0, t = 1$ respectively), and after computing the reduced column echelon form and looking for missing leading 1's, we obtain a relative basis consisting of the vector $f = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. This vector gives rise to the thread f, $Af = \begin{pmatrix} 8 \\ 23 \\ -1 \end{pmatrix}$, $A^2 f = \begin{pmatrix} -1 \\ -3 \\ 0 \end{pmatrix}$. Since our space is 3-dimensional, this thread forms a basis.

Example 4. $V = \mathbb{R}^4$, $A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & -1 \end{pmatrix}$.

In this case, $A^2 = 0$, $\operatorname{rk}(A) = 2$, $\operatorname{rk}(A^k) = 0$ for $k \geqslant 2$, $\dim \operatorname{Ker}(A) = 2$, $\dim \operatorname{Ker}(A^k) = 4$ for $k \geqslant 2$. Moreover, $\operatorname{Ker}(A) = \{\begin{pmatrix} -s \\ t \\ t \\ s \end{pmatrix}\}$.

We have a sequence of subspaces $V = \operatorname{Ker}(A^2) \supset \operatorname{Ker}(A) \supset \{0\}$. The first one relative to the second one is two-dimensional ($\dim \operatorname{Ker}(A^2) - \dim \operatorname{Ker}(A) = 2$). Clearly, the vectors $\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ (corresponding to $s = 1, t = 0$ and $s = 0, t = 1$ respectively) form a basis of the kernel of A, and after computing the reduced column echelon form and looking for missing leading 1's, we obtain a relative basis consisting of the vectors $f_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $f_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$. These vectors give rise to threads f_1, $Af_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ and f_2, $Af_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$. These two threads together contain four vectors, and we have a basis.