1111: Linear Algebra I

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Lecture 12

Previously on. . .

Let us consider the following question:

> *Given some numbers in the first row, the last row, the first column, and the last column of an $n \times n$-matrix, is it possible to fill the numbers in all the remaining slots in a way that each of them is the average of its 4 neighbours?*

This is the "discrete Dirichlet problem", a finite grid approximation to many foundational questions of mathematical physics.

An example for the Fredholm alternative

For instance, for $n = 4$ we may face the following problem: find a, b, c, d to put in the matrix

$$\begin{pmatrix} 4 & 3 & 0 & 1.5 \\ 1 & a & b & -1 \\ 0.5 & c & d & 2 \\ 2.1 & 4 & 2 & 1 \end{pmatrix}$$

so that

$$\begin{cases} a = \frac{1}{4}(3 + 1 + b + c), \\ b = \frac{1}{4}(a + 0 - 1 + d), \\ c = \frac{1}{4}(a + 0.5 + d + 4), \\ d = \frac{1}{4}(b + c + 2 + 2). \end{cases}$$

This is a system with 4 equations and 4 unknowns.

An example for the Fredholm alternative

In general, we shall be dealing with a system with $(n-2)^2$ equations and $(n-2)^2$ unknowns.

Note that according to the Fredholm alternative, it is enough to prove that for the zero boundary data we get just the trivial solution. Let a_{ij} be a solution for the zero boundary data. Let a_{PQ} be the largest element among them. Since

$$a_{PQ} = \frac{1}{4}(a_{P-1,Q} + a_{P,Q-1} + a_{P+1,Q} + a_{P,Q+1} \leqslant \frac{1}{4}(a_{PQ} + a_{PQ} + a_{PQ} + a_{PQ}),$$

the neighbours of a_{PQ} must all be equal to a_{PQ}. Similarly, their neighbours must be equal to a_{PQ} etc., and it propagates all the way to the boundary, so we observe that $a_{PQ} = 0$. The same argument appliest with the smallest element, and we conclude that all elements must be equal to zero. This, as we already realised, proves that for every choice of the boundary data the solution is unique.

Summary of systems of linear equations

For systems of equations where the number of equations is not necessarily equal to the number of unknowns, there is one key result that we shall use extensively in the further parts of the module.

> *A homogeneous system $Ax = 0$ with n unknowns and $m < n$ equations always has a nontrivial solution.*

The proof is completely trivial. Indeed, there will be no inconsistencies of the type $0 = 1$, and there will be at least one free unknown since $m < n$.

Application: Let A be an $n \times n$-matrix. Then there exist k and some numbers $c_0, c_1, \ldots, c_{k-1}$ such that

$$A^k = c_{k-1}A^{k-1} + \cdots + c_1 A + c_0 I_n \ .$$

In fact, can take $k = n^2$. (Indeed, $c_k A^k + c_{k-1}A^{k-1} + \cdots + c_1 A + c_0 I_n = 0$ is a homogeneous system with $k + 1$ unknowns and n^2 equations...)

An application of determinants: Vandermonde determinant

Let $x_1, \ldots, x_n$ be scalars. The *Vandermonde determinant* $V(x_1, \ldots, x_n)$ is the determinant of the matrix

$$\begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ x_1 & x_2 & x_3 & \ldots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \ldots & x_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \ldots & x_n^{n-1} \end{pmatrix}.$$

Theorem. We have

$$V(x_1, \ldots, x_n) =$$
$$(x_2 - x_1)(x_3 - x_2)(x_3 - x_1) \cdots (x_n - x_{n-1}) = \prod_{i<j} (x_j - x_i).$$

The Vandermonde determinant

Theorem. We have

$$V(x_1,\ldots,x_n) = \\ (x_2-x_1)(x_3-x_2)(x_3-x_1)\cdots(x_n-x_{n-1}) = \prod_{i<j}(x_j-x_i).$$

Proof: Let us subtract, for each $i = n-1, n-2, \ldots, 1$, the row i times x_1 from the row $i+1$. Combining rows does not change the determinant, so we conclude that $V(x_1,\ldots,x_n)$ is equal to the determinant of the matrix

$$\begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 0 & x_2-x_1 & x_3-x_1 & \ldots & x_n-x_1 \\ 0 & x_2^2-x_1x_2 & x_3^2-x_1x_3 & \ldots & x_n^2-x_1x_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & x_2^{n-1}-x_1x_2^{n-2} & x_3^{n-1}-x_1x_3^{n-2} & \ldots & x_n^{n-1}-x_1x_n^{n-2} \end{pmatrix}.$$

The Vandermonde determinant

Let us expand the determinant

$$\det\begin{pmatrix}1 & 1 & 1 & \dots & 1\\ 0 & x_2-x_1 & x_3-x_1 & \dots & x_n-x_1\\ 0 & x_2^2-x_1x_2 & x_3^2-x_1x_3 & \dots & x_n^2-x_1x_n\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & x_2^{n-1}-x_1x_2^{n-2} & x_3^{n-1}-x_1x_3^{n-2} & \dots & x_n^{n-1}-x_1x_n^{n-2}\end{pmatrix}$$

along the first column, the result is

$$\det\begin{pmatrix}x_2-x_1 & x_3-x_1 & \dots & x_n-x_1\\ x_2^2-x_1x_2 & x_3^2-x_1x_3 & \dots & x_n^2-x_1x_n\\ \vdots & \vdots & \ddots & \vdots\\ x_2^{n-1}-x_1x_2^{n-2} & x_3^{n-1}-x_1x_3^{n-2} & \dots & x_n^{n-1}-x_1x_n^{n-2}\end{pmatrix}.$$

The Vandermonde determinant

We note that the k-th column of the determinant

$$\det \begin{pmatrix} x_2 - x_1 & x_3 - x_1 & \ldots & x_n - x_1 \\ x_2^2 - x_1 x_2 & x_3^2 - x_1 x_3 & \ldots & x_n^2 - x_1 x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_2^{n-1} - x_1 x_2^{n-2} & x_3^{n-1} - x_1 x_3^{n-2} & \ldots & x_n^{n-1} - x_1 x_n^{n-2} \end{pmatrix}$$

is divisible by $x_{k+1} - x_1$, so it is equal to

$$(x_2 - x_1)(x_3 - x_1) \cdots (x_n - x_1) \det \begin{pmatrix} 1 & 1 & \ldots & 1 \\ x_2 & x_3 & \ldots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_2^{n-2} & x_3^{n-2} & \ldots & x_n^{n-2} \end{pmatrix},$$

so we encounter a smaller Vandermonde determinant, and can proceed by induction. $\square$

The Vandermonde determinant

Another (sneaky) proof: we see that $V(x_1,\ldots,x_n)=0$ whenever $x_i=x_j$ for $i\neq j$ (two equal columns). Therefore, the polynomial expression $V(x_1,\ldots,x_n)$ is divisible by all x_i-x_j for $i>j$. But the degree of $V(x_1,\ldots,x_n)$ is $1+2+\cdots+n-1=\frac{n(n-1)}{2}$ (because we take one element from each row), and the degree of the product

$$(x_2-x_1)(x_3-x_2)(x_3-x_1)\cdots(x_n-x_{n-1})$$

is $1+2+\cdots+n-1$, so these polynomial expression differ by a scalar multiple. Comparing the coefficients of $x_2x_3^2\cdots x_n^{n-1}$ (the diagonal term), we find that both coefficients are 1, so there is an equality. $\square$

There are some “gaps” that are not hard to fill but need to be filled. Those who take the module 2215 next year, will be able to complete the proof formally, others need some trust.

The Vandermonde determinant

An important consequence: the Vandermonde determinant is not equal to zero if and only if $x_1, \ldots, x_n$ are all distinct.

Theorem. For each n distinct numbers $x_1, \ldots, x_n$, and each choice of $a_1, \ldots, a_n$, there exists a unique polynomial $f(x) = c_0 + c_1 x + \cdots + c_{n-1}x^{n-1}$ of degree at most $n-1$ such that $f(x_1) = a_1, \ldots, f(x_n) = a_n$.

Proof: Let us figure out what conditions are imposed on the coefficients $c_0, \ldots, c_{n-1}$:

$$\begin{cases} c_0 + c_1 x_1 + \cdots + c_{n-1}x_1^{n-1} = a_1, \\ c_0 + c_1 x_2 + \cdots + c_{n-1}x_2^{n-1} = a_2, \\ \ldots, \\ c_0 + c_1 x_n + \cdots + c_{n-1}x_n^{n-1} = a_n \ . \end{cases}$$

The matrix of this system is the transpose of the Vandermonde matrix!

The Vandermonde determinant

We conclude that the conditions we wish to observe are of the form $Ax = b$, where $b = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$ and $\det(A) = V(x_1 \ldots, x_n)$. Since $x_1, \ldots, x_n$ are distinct, $\det(A) \neq 0$, and the system has exactly one solution for any choice of the vector b. $\square$

Remark. In fact, one can write the formula for $f(x)$ directly. The following neat formula for $f(x)$ is called the *Lagrange interpolation formula*:

$$f(x) = \sum_{i=1}^{n} a_i \frac{(x - x_1) \cdots (x - x_{i-1})(x - x_{i+1}) \cdots (x - x_n)}{(x_i - x_1) \cdots (x_i - x_{i-1})(x_i - x_{i+1}) \cdots (x_i - x_n)} .$$

The conditions $f(x_i) = a_i$ are easily checked by inspection.