1111: Linear Algebra I

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Lecture 18

Dimension: examples

Example 1. The dimension of $\mathbb{R}^n$ is equal to n, as expected. (Standard unit vectors form a basis).

Example 2. The dimension of the space of polynomials in one variable x of degree at most n is equal to $n+1$, since it has a basis $1, x, \ldots, x^n$.

Example 3. The dimension of the space of $m \times n$-matrices is equal to mn. (Matrix units e_{ij}, that is matrices that have the only nonzero element equal to 1, which is at the intersection of the i-th row and the j-th column, form a basis).

Example 4. For a matrix A, the dimension of the solution space to the system of equations $Ax = 0$ is equal to the number of free unknowns, that is the number of columns of the reduced row echelon form of A that do not have pivots. (The spanning set we constructed previously forms a basis).

We also discussed in detail one of the tutorial questions — see the handout for the tutorial for that solution.

Change of coordinates

Let V be a vector space of dimension n, and let $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ be two different bases of V. Then we can compute coordinates of each vector v with respect to either of those bases, so that

$$v = x_1 e_1 + \cdots + x_n e_n$$

and

$$v = y_1 f_1 + \cdots + y_n f_n.$$

Our goal now is to figure out how these are related. For that, we shall need the notion of a transition matrix.

Definition 1. Let us express the vectors $f_1, \ldots, f_n$ as linear combinations of $e_1, \ldots, e_n$:

$$\begin{aligned}
f_1 &= a_{11} e_1 + a_{21} e_2 + \cdots + a_{m1} e_m, \\
f_2 &= a_{12} e_1 + a_{22} e_2 + \cdots + a_{m2} e_m, \\
&\ldots \\
f_n &= a_{1n} e_1 + a_{2n} e_2 + \cdots + a_{mn} e_m.
\end{aligned}$$

The matrix (a_{ij}) is called *the transition matrix* from the basis $e_1, \ldots, e_n$ to the basis $f_1, \ldots, f_n$. Its k-th column is the column of coordinates of the vector f_k relative to the basis $e_1, \ldots, e_n$.

Lemma 1. *In the notation above, we have*

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \dots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}.$$

In plain words, if we call $e_1, \ldots, e_n$ the "old basis" and $f_1, \ldots, f_n$ the "new basis", then this system tells us that the product of the transition matrix with the columns of new coordinates of a vector is equal to the column of old coordinates.

Proof. The proof is fairly straightforward: we take the formula

$$v = y_1 f_1 + \cdots + y_n f_n,$$

and substitute instead of f_i's their expressions in terms of e_j's:

$$f_1 = a_{11}e_1 + a_{21}e_2 + \cdots + a_{m1}e_m,$$
$$f_2 = a_{12}e_1 + a_{22}e_2 + \cdots + a_{m2}e_m,$$
$$\dots$$
$$f_n = a_{1n}e_1 + a_{2n}e_2 + \cdots + a_{mn}e_m.$$

What we get is

$$y_1(a_{11}e_1+a_{21}e_2+\cdots+a_{n1}e_n)+y_2(a_{12}e_1+a_{22}e_2+\cdots+a_{n2}e_n)+\ldots+y_n(a_{1n}e_1+a_{2n}e_2+\cdots+a_{nn}e_n) =$$
$$= (a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n)e_1 + \cdots + (a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n)e_n.$$

Since we know that coordinates are uniquely defined, we conclude that

$$a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n = x_1,$$
$$\dots$$
$$a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n = x_n,$$

which is what we want to prove. $\square$

If we denote, for a vector v, the column of coordinates of v with respect to the basis $e_1, \ldots, e_n$ by $v_{\mathbf{e}}$, and also denote the transition matrix from the basis $e_1, \ldots, e_n$ to the basis $f_1, \ldots, f_n$ by $M_{\mathbf{e},\mathbf{f}}$, then the previous result can be written as

$$v_{\mathbf{e}} = M_{\mathbf{e},\mathbf{f}} v_{\mathbf{f}}.$$

Lemma 2. *We have*

$$M_{\mathbf{e},\mathbf{f}} M_{\mathbf{f},\mathbf{g}} = M_{\mathbf{e},\mathbf{g}}$$

and

$$M_{\mathbf{e},\mathbf{f}} M_{\mathbf{f},\mathbf{e}} = I_n$$

if $\dim(V) = n$.

Proof. Applying the formula above twice, we have

$$v_{\mathbf{e}} = M_{\mathbf{e},\mathbf{f}} v_{\mathbf{f}} = M_{\mathbf{e},\mathbf{f}} M_{\mathbf{f},\mathbf{g}} v_{\mathbf{g}}.$$

But we also have

$$v_{\mathbf{e}} = M_{\mathbf{e},\mathbf{g}} v_{\mathbf{g}}.$$

Therefore

$$M_{\mathbf{e},\mathbf{f}} M_{\mathbf{f},\mathbf{g}} v_{\mathbf{g}} = M_{\mathbf{e},\mathbf{g}} v_{\mathbf{g}}$$

for every $v_{\mathbf{g}}$. From our previous classes we know that knowing $A\mathbf{v}$ for all vectors $\mathbf{v}$ completely determines the matrix A, so $M_{\mathbf{e},\mathbf{f}} M_{\mathbf{f},\mathbf{g}} = M_{\mathbf{e},\mathbf{g}}$ as required. Since manifestly we have $M_{\mathbf{e},\mathbf{e}} = I_n$, we conclude by letting $g_k = e_k$, $k = 1, \ldots, n$, that $M_{\mathbf{e},\mathbf{f}} M_{\mathbf{f},\mathbf{e}} = I_n$. $\square$