1111: Linear Algebra I

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Lecture 20

Linear maps and change of coordinates

As a last step, let us exhibit how matrices of linear maps transform under changes of coordinates.

Lemma 1. *Let* $\varphi\colon V \to W$ *be a linear operator, and suppose that* $e_1, \ldots, e_n$ *and* $e'_1, \ldots, e'_n$ *are two bases of* V*, and* $f_1, \ldots, f_m$ *and* $f'_1, \ldots, f'_m$ *are two bases of* W*. Then*

$$A_{\varphi,\mathbf{e}',\mathbf{f}'} = M_{\mathbf{f}',\mathbf{f}} A_{\varphi,\mathbf{e},\mathbf{f}} M_{\mathbf{e},\mathbf{e}'} = M^{-1}_{\mathbf{f},\mathbf{f}'} A_{\varphi,\mathbf{e},\mathbf{f}} M_{\mathbf{e},\mathbf{e}'}.$$

Proof. Let us take a vector $\mathbf{v} \in V$. On the one hand, the formula of Lemma 2 tells us that

$$(\varphi(\mathbf{v}))_{\mathbf{f}'} = A_{\varphi,\mathbf{e}',\mathbf{f}'} \mathbf{v}_{\mathbf{e}'}.$$

On the other hand, applying various results we proved earlier, we have

$$(\varphi(\mathbf{v}))_{\mathbf{f}'} = M_{\mathbf{f}',\mathbf{f}}(\varphi(\mathbf{v})_{\mathbf{f}}) = M_{\mathbf{f}',\mathbf{f}}(A_{\varphi,\mathbf{e},\mathbf{f}} \mathbf{v}_{\mathbf{e}}) = M_{\mathbf{f}',\mathbf{f}}(A_{\varphi,\mathbf{e},\mathbf{f}}(M_{\mathbf{e},\mathbf{e}'} \mathbf{v}_{\mathbf{e}'})) = (M_{\mathbf{f}',\mathbf{f}} A_{\varphi,\mathbf{e},\mathbf{f}} M_{\mathbf{e},\mathbf{e}'}) \mathbf{v}_{\mathbf{e}'}.$$

Therefore,

$$A_{\varphi,\mathbf{e}',\mathbf{f}'} \mathbf{v}_{\mathbf{e}'} = (M_{\mathbf{f}',\mathbf{f}} A_{\varphi,\mathbf{e},\mathbf{f}} M_{\mathbf{e},\mathbf{e}'}) \mathbf{v}_{\mathbf{e}'}$$

for every $\mathbf{v}_{\mathbf{e}'}$. From our previous classes we know that knowing $A\mathbf{v}$ for all vectors $\mathbf{v}$ completely determines the matrix A, so

$$A_{\varphi,\mathbf{e}',\mathbf{f}'} = (M_{\mathbf{f}',\mathbf{f}} A_{\varphi,\mathbf{e},\mathbf{f}} M_{\mathbf{e},\mathbf{e}'}) = (M^{-1}_{\mathbf{f},\mathbf{f}'} A_{\varphi,\mathbf{e},\mathbf{f}} M_{\mathbf{e},\mathbf{e}'})$$

because of properties of transition matrices proved earlier. □

Remark 1. Our formula

$$A_{\varphi,\mathbf{e}',\mathbf{f}'} = M_{\mathbf{f}',\mathbf{f}} A_{\varphi,\mathbf{e},\mathbf{f}} M_{\mathbf{e},\mathbf{e}'}.$$

shows that changing from the coordinate systems $\mathbf{e}, \mathbf{f}$ to *some* other coordinate system amounts to multiplying the matrix $A_{\varphi,\mathbf{e},\mathbf{f}}$ by some invertible matrices on the left and on the right, so effectively to performing a certain number of elementary row and column operations on this matrix. This is very useful (but not applicable to a more narrow class of linear transformations, see below).

Remark 2. A linear operator $\varphi\colon V \to V$ is often called a *linear transformation.* For a linear transformation, it makes sense to use the same coordinate system for the input and the output. By definition, the matrix of a linear operator $\varphi\colon V \to V$ relative to the basis $e_1, \ldots, e_n$ is

$$A_{\varphi,\mathbf{e}} := A_{\varphi,\mathbf{e},\mathbf{e}}.$$

Lemma 2. *For a linear transformation* $\varphi\colon V \to V$*, and two bases* $e_1, \ldots, e_n$ *and* $e'_1, \ldots, e'_n$ *of* V*, we have*

$$A_{\varphi,\mathbf{e}'} = M^{-1}_{\mathbf{e},\mathbf{e}'} A_{\varphi,\mathbf{e}} M_{\mathbf{e},\mathbf{e}'}.$$

Proof. This is a particular case of Lemma 1. □

Remark 3. Proposition 5 shows that for a square matrix A, the change $A \mapsto C^{-1}AC$ with an invertible matrix C, corresponds to the situation where A is viewed as a matrix of a linear transformation, and C is viewed as a transition matrix for a coordinate change. You verified in your earlier home assignments that $\mathrm{tr}(C^{-1}AC) = \mathrm{tr}(A)$ and $\det(C^{-1}AC) = \det(A)$; these properties imply that the trace and the determinant do not depend on the choice of coordinates, and hence reflect some geometric properties of a linear transformation. (In case of the determinant, those properties have been hinted at in our previous classes: determinants compute how a linear transformation changes volumes of solids).

Examples of linear maps and coordinate changes

Example 1. As we know, every linear map $\varphi\colon \mathbb{R}^n \to \mathbb{R}^k$ is given by a $k \times n$-matrix A, so that $\varphi(x) = Ax$.

Example 2. Let V be the vector space of all polynomials in one variable x. Consider the function $\varphi\colon V \to V$ that maps every polynomial $f(x)$ to $xf(x)$. This is a linear map:

$$x(f_1(x) + f_2(x)) = xf_1(x) + xf_2(x),$$
$$x(cf(x)) = c(xf(x)).$$

Example 3. Let V be the vector space of all polynomials in one variable x. Consider the function $\psi\colon V \to V$ that maps every polynomial $f(x)$ to $f'(x)$. This is a linear map:

$$(f_1(x) + f_2(x))' = f_1'(x) + f_2'(x),$$
$$(cf(x))' = cf'(x).$$

Example 4. Let V be the vector space of all polynomials in one variable x. Consider the function $\alpha\colon V \to V$ that maps every polynomial $f(x)$ to $3f(x)f'(x)$. This is not a linear map; for example, $1 \mapsto 0$, $x \mapsto 3x$, but $x + 1 \mapsto 3(x + 1) = 3x + 3 \neq 3x + 0$.