1111: Linear Algebra I

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Lecture 21

Examples of linear maps and coordinate changes

Example 1. Let P_n be the vector space of all polynomials in one variable x of degree at most n. Then there is a function $\varphi\colon P_n \to P_{n+1}$ that maps every polynomial $f(x)$ to $xf(x)$. (Note that the target of φ has to be different, since multiplying by x increases degrees). This function is a linear map, which we can check in the same way as we did in previous class.

Example 2. Let P_n be the vector space of all polynomials in one variable x of degree at most n. Then we can define both a function $\psi\colon P_n \to P_{n-1}$ that maps every polynomial $f(x)$ to $f'(x)$, and a function $\hat{\psi}\colon P_n \to P_n$ that every polynomial $f(x)$ to $f'(x)$ (since the degree of the derivative of a polynomial of degree *at most* n is *at most* $n-1$). These functions are linear maps, which we can check in the same way as in previous class. In fact, $\hat{\psi}$ is a linear transformation, since it is a map from P_n to itself.

Example 3. Consider the vector space M_2 of all 2×2-matrices. Let us define a function $\alpha\colon M_2 \to M_2$ by the formula $\alpha(X) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} X$. Let us check that this map is a linear transformation. Indeed, by properties of matrix products

$$\alpha(X_1 + X_2) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}(X_1+X_2) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}X_1 + \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}X_2 = \alpha(X_1)+\alpha(X_2),$$
$$\alpha(cX) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}(cX) = c\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}X = c\alpha(X).$$

Example 4. Let us consider the linear map φ from Example 1, and assume $n=2$. Let us take the bases $e_1 = 1, e_2 = x, e_3 = x^2$ of P_2, and the basis $f_1 = 1, f_2 = x, f_3 = x^2, f_4 = x^3$ of P_3, and compute $A_{\varphi,\mathbf{e},\mathbf{f}}$. Note that $\varphi(e_1) = x\cdot 1 = x = f_2$, $\varphi(e_2) = x\cdot x = x^2 = f_3$, and $\varphi(e_3) = x\cdot x^2 = x^3 = f_4$. Therefore

$$A_{\varphi,\mathbf{e},\mathbf{f}} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Example 5. Let us consider the linear maps ψ and $\hat{\psi}$ from Example 2, and assume $n=3$. Let us take the bases $e_1 = 1, e_2 = x, e_3 = x^2, e_4 = x^3$ of P_3, and the basis $f_1 = 1, f_2 = x, f_3 = x^2$ of P_2, and let us compute $A_{\psi,\mathbf{e},\mathbf{f}}$ and $A_{\hat{\psi},\mathbf{e}}$. Note that $\psi(e_1) = 1' = 0$, $\psi(e_2) = x' = 1 = f_1$, $\psi(e_3) = (x^2)' = 2x = 2f_2$, and $\psi(e_4) = (x^3)' = 3x^2 = 3f_3$, and that $\hat{\psi}(e_1) = 1' = 0$, $\hat{\psi}(e_2) = x' = 1 = e_1$, $\hat{\psi}(e_3) = (x^2)' = 2x = 2e_2$, and $\hat{\psi}(e_4) = (x^3)' = 3x^2 = 3e_3$. Therefore

$$A_{\psi,\mathbf{e},\mathbf{f}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

and

$$A_{\hat{\psi},\mathbf{e}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Example 6. Let us look at the linear map α from Example 3. We consider the basis of matrix units in M_2: $e_1 = E_{11}$, $e_2 = E_{12}$, $e_3 = E_{21}$, $e_4 = E_{22}$. We have

$$\begin{aligned}
\alpha(e_1) &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = e_1 + e_3, \\
\alpha(e_2) &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} e_2 + e_4, \\
\alpha(e_3) &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = e_1, \\
\alpha(e_4) &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = e_2,
\end{aligned}$$

so

$$A_{\alpha,\mathbf{e}} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Example 7. Let us take two bases of $\mathbb{R}^2$: $e_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $e_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and $f_1 = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$, $f_2 = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$. Suppose that the matrix of a linear transformation $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ relative to the first basis is $\begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}$. Let us compute its matrix relative to the second basis. For that, we first compute the transition matrix $M_{\mathbf{e},\mathbf{f}}$. We have

$$\begin{aligned}
f_1 &= \begin{pmatrix} 7 \\ 5 \end{pmatrix} = 5e_1 + 2e_2, \\
f_2 &= \begin{pmatrix} 4 \\ 3 \end{pmatrix} = 3e_1 + e_2,
\end{aligned}$$

so

$$M_{\mathbf{e},\mathbf{f}} = \begin{pmatrix} 5 & 3 \\ 2 & 1 \end{pmatrix},$$

and

$$M_{\mathbf{e},\mathbf{f}}^{-1} = \begin{pmatrix} -1 & 3 \\ 2 & -5 \end{pmatrix}.$$

Therefore

$$A_{\varphi,\mathbf{f}} = M_{\mathbf{e},\mathbf{f}}^{-1} A_{\varphi,\mathbf{e}} M_{\mathbf{e},\mathbf{f}} = \begin{pmatrix} -1 & 3 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 5 & 3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} -2 & -3 \\ 10 & 9 \end{pmatrix}.$$

Computing Fibonacci numbers

Fibonacci numbers are defined recursively: $f_0 = 0$, $f_1 = 1$, $f_n = f_{n-1} + f_{n-2}$ for $n \geqslant 2$, so that this sequence starts like this:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$$

Next time we shall discuss how to derive a formula for these using linear algebra.