1111: Linear Algebra I

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Lecture 7

Invertible matrices

Theorem. 1. An elementary matrix is invertible.
2. If an $n \times m$-matrix A is invertible, then $m = n$.
3. An $n \times n$-matrix A is invertible if and only if it can be represented as a product of elementary matrices.

Proof. 1. Proved in the previous class.
2. Suppose that $m \neq n$, and there exist matrices A and B such that $A \cdot B = I_m$ and $B \cdot A = I_n$. Without loss of generality, $m > n$ (otherwise swap A with B). Let us show that $AB = I_m$ leads to a contradiction. We have $E_1 \cdot E_2 \cdots E_p \cdot A = R$, where R is the reduced row echelon form of A, and E_i are appropriate elementary matrices. Therefore,

$$R \cdot B = E_1 \cdot E_2 \cdots E_p \cdot A \cdot B = E_1 \cdot E_2 \cdots E_p \ .$$

From that, we immediately deduce

$$R \cdot B \cdot (E_p)^{-1} \cdots (E_2)^{-1} \cdot (E_1)^{-1} = I_m \ .$$

Invertible matrices

But if we assume $m > n$, the last row of R is inevitably zero (there is no room for m pivots), so the last row of

$$I_m = R \cdot B \cdot (E_p)^{-1} \cdots (E_2)^{-1} \cdot (E_1)^{-1}$$

is zero too, a contradiction.

3. If A can be represented as a product of elementary matrices, it is invertible, since products of invertible matrices are invertible. If A is invertible, then the last row of its reduced row echelon form must be non-zero, or we get a contradiction like in the previous argument. Therefore, each row of the reduced row echelon form of A, and hence, by previous result, each column of the reduced row echelon form of A, has a pivot, so the reduced row echelon form of A is the identity matrix. We conclude that $E_1 \cdot E_2 \cdots E_p \cdot A = I_n$, so $A = (E_p)^{-1} \cdots (E_2)^{-1} \cdot (E_1)^{-1}$, which is a product of elementary matrices. $\square$

One more property of inverses

There is another useful property that is proved completely analogously:

> *If for an $n \times n$-matrix A, there exists a "one-sided" inverse (that is, B for which only one of the two conditions $AB = I_n$ and $BA = I_n$ are satisfied), then $B = A^{-1}$.*

To prove it, it is enough to consider the case $AB = I_n$ (otherwise we can swap the roles of A and B). In this case, we proceed as before to conclude that the reduced row echelon form of A cannot have a row of zeros, hence that reduced row echelon form is the identity matrix, hence A is invertible. Finally, $A^{-1}(AB) = (A^{-1}A)B = I_nB = B$.

Warning: we know that for $m \neq n$ an $m \times n$-matrix cannot be invertible, but such a matrix can have a one-sided inverse. You will be asked to construct an example in the next homework.

Computing inverses

Our results lead to an elegant algorithm for computing the inverse of an $n \times n$-matrix A.

Form an $n \times (2n)$-matrix $(A \mid I_n)$. Apply the usual algorithm to compute its reduced row echelon form. If A is invertible, the output is a matrix of the form $(I_n \mid B)$, where $B = A^{-1}$.

Justification. If A is invertible, its reduced row echelon form is the identity matrix I_n. Therefore, the computation of the reduced row echelon form of $(A \mid I_n)$ will produce a matrix of the form $(I_n \mid B)$, since pivots emerge from the left to the right. This matrix is clearly in its reduced row echelon form. Let us take the elementary matrices corresponding to the appropriate row operations, so that $E_1 \cdot E_2 \cdots E_p \cdot A = I_n$. This means, as we just proved, that $A^{-1} = E_1 \cdot E_2 \cdots E_p$. It remains to remark that

$$E_1 \cdot E_2 \cdots E_p \cdot (A \mid I_n) = (E_1 \cdot E_2 \cdots E_p \cdot A \mid E_1 \cdot E_2 \cdots E_p),$$

so $(I_n \mid B) = (I_n \mid E_1 \cdot E_2 \cdots E_p) = (I_n \mid A^{-1})$. $\square$

Towards computing determinants

In school, you may have seen either the formula

$$A^{-1} = \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$$

for a 2×2-matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, or its consequence, the formula

$$\begin{cases} x = \frac{de-bf}{ad-bc}, \\ y = \frac{af-ce}{ad-bc}, \end{cases}$$

allowing to solve a system of two equations with two unknowns

$$\begin{cases} ax + by = e, \\ cx + dy = f. \end{cases}$$

Now, we shall see how to generalise these formulas for $n \times n$-matrices.

Permutations

To proceed, we need to introduce the notion of a *permutation*. By definition, a permutation of n elements is a rearrangement of numbers $1, 2, \dots, n$ in a particular order.

For example, $1, 3, 4, 2$ is a permutation of four elements, and $1, 4, 3, 4, 2$ is not (because the number 4 is repeated).

We shall also use the two-row notation for permutations: a permutation of n elements may be represented by a $2 \times n$-matrix made up of columns $\begin{pmatrix} j \\ a_j \end{pmatrix}$, where a_j is the number at the j-th place in the permutation.

For example, the permutation $1, 3, 4, 2$ may be represented by the matrix $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}$, but also by the matrix $\begin{pmatrix} 1 & 4 & 3 & 2 \\ 1 & 2 & 4 & 3 \end{pmatrix}$, and by many other matrices.

Incidentally, the number of different permutations of n elements is equal to $1 \cdot 2 \cdots n$, this number is called "n factorial" and is denoted by $n!$ (n with an exclamation mark).

Odd and even permutations

Let σ be a permutation of n elements, written in the one-row notation. Two numbers i and j, where $1 \leqslant i < j \leqslant n$, are said to form an inversion in σ, if they are "listed in wrong order", that is j appears before i in σ.

For the permutation $1, 3, 4, 2$, there are 6 pairs (i, j) to look at: $(1, 2)$, $(1, 3)$, $(1, 4)$, $(2, 3)$, $(2, 4)$, $(3, 4)$. Of these, the pair $(2, 3)$ forms an inversion, and the pair $(2, 4)$ does, and other pairs do not.

A permutation is said to be *even* if its number of inversions is even, and *odd* otherwise. One of the most important properties of this division into even and odd (which we shall prove next week) is the following: *if we swap two numbers in a permutation $a_1, \ldots, a_n$, it makes an even permutation into an odd one, and vice versa*.