1212: Linear Algebra II

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Lecture 11

Bilinear and quadratic forms

Recall the definition from the last class.

Definition 1. Let V be a vector space. A function $V \times V \to \mathbb{R}$, $v_1, v_2 \mapsto b(v_1, v_2)$ is called a bilinear form if for all vectors v, v_1, v_2 the following conditions are satisfied:

$$b(c_1v_1 + c_2v_2, v) = c_1b(v_1, v) + c_2b(v_2, v) \quad \text{and} \quad b(v, c_1v_1 + c_2v_2) = c_1b(v, v_1) + c_2b(v, v_2).$$

A bilinear form is said to be *symmetric* if $b(v_1, v_2) = b(v_2, v_1)$ for all v_1, v_2, and *skew-symmetric* if $b(v_1, v_2) = -b(v_2, v_1)$ for all v_1, v_2. A symmetric bilinear form is said to be *positive semidefinite* if $b(v, v) \geqslant 0$ for all v, and *positive definite*, if $b(v, v) > 0$ for $v \neq 0$. In these words, a function of two vector arguments is a scalar product if and only if it is bilinear, symmetric, and positive definite.

Remark 1. Generalising what we proved about scalar products, for every bilinear form b and every basis $e_1, \ldots, e_n$ of V, we have

$$b(x_1e_1 + \ldots + x_ne_n, y_1e_1 + \ldots + y_ne_n) = \sum_{i,j=1}^{n} b_{ij}x_iy_j,$$

where $b_{ij} = b(e_i, e_j)$. Moreover, this number corresponds to the 1×1-matrix x^TBy, where B is the matrix with entries b_{ij}.

Every bilinear form b gives rise to a quadratic form by putting $q(x) = b(x, x)$, for example, the bilinear form

$$b(x_1e_1 + x_2e_2, y_1e_1 + y_2e_2) = 2x_1y_2$$

gives rise to a quadratic form $2x_1x_2$, and the bilinear form

$$b(x_1e_1 + x_2e_2, y_1e_1 + y_2e_2) = x_1y_2 + x_2y_1$$

gives rise to the same quadratic form. It turns out that the reconstruction of b from q is unique if we assume that b is symmetric; in this case the reconstruction formula is

$$b(v, w) := \frac{1}{2}(q(v + w) - q(v) - q(w)).$$

Indeed, if $q(v) = b(v, v)$, then

$$\frac{1}{2}(q(v + w) - q(v) - q(w)) = \frac{1}{2}(b(v + w, v + w) - b(v, v) - b(w, w)) =$$
$$= \frac{1}{2}(b(v, v) + b(v, w) + b(w, v) + b(w, w) - b(v, v) - b(w, w)) = \frac{1}{2}(b(v, w) + b(w, v)),$$

which, for a symmetric bilinear form, is $b(v, w)$.

The coefficients a_{ij} of a quadratic form and the coefficients b_{ij} of the corresponding symmetric bilinear form are related by $a_{ii} = b_{ii}$ and $a_{ij} = b_{ij} + b_{ji} = 2b_{ij}$ for $i < j$.

We shall now formulate several theorems about quadratic forms and symmetric bilinear forms; they will be a topic of our tutorial class and the next problem sheet, and next week we shall discuss their proofs in detail.

One celebrated example of a quadratic form is $q(x_1, x_2, x_3, t) = x_1^2 + x_2^2 + x_3^2 - t^2$ on the Minkowski space $\mathbb{R}^4$, it is used in special relativity theory. This serves as a (humble) motivation for the following result.

Theorem 1. *Let q be a quadratic form on a vector space V. There exists a basis $f_1, \ldots, f_n$ of V for which the quadratic form q becomes a signed sum of squares:*

$$q(x_1 f_1 + \cdots + x_n f_n) = \sum_{i=1}^{n} \varepsilon_i x_i^2,$$

where all numbers ε_i are either 1 or -1 or 0.

Theorem 2 (Law of inertia)**.** *In the previous theorem, the triple (n_+, n_-, n_0), where $n_\pm$ is the number of ε_i equal to ± 1, and n_0 is the number of ε_i equal to 0, does not depend on the choice of the basis $f_1, \ldots, f_n$. This triple is often referred to as the* signature *of the quadratic form q.*

Let $B = (b_{ij})$ be the matrix of a given symmetric bilinear form b on V. We shall now discuss some methods of computing the signature of b via the matrix elements of B.

Theorem 3. *The signature of B is completely determined by eigenvalues of B: the number n_+ is the number of positive eigenvalues, the number n_- is the number of negative eigenvalues, and the number n_0 is the number of zero eigenvalues.*

Note that this theorem makes sense because all eigenvalues of a symmetric matrix are real.

Let us denote by B_k the $k \times k$-matrix whose entries are b_{ij} with $1 \leqslant i, j \leqslant k$, that is the top left corner submatrix of B. We put $\Delta_0 = 1$ and $\Delta_k := \det(B_k)$ for $1 \leqslant k \leqslant n$.

Theorem 4 (Jacobi diagonal form)**.** *Suppose that for all $i = 1, \ldots, n$ we have $\Delta_i \neq 0$. Then there exists a system of coordinates where the matrix of b is a diagonal matrix with the numbers $\frac{\Delta_{k-1}}{\Delta_k}$ on the diagonal.*

Theorem 5 (Sylvester's criterion)**.** *The given symmetric bilinear form is positive definite if and only if*

$$\Delta_k > 0 \quad \textit{for all} \quad k = 1, \ldots, n.$$