1212: Linear Algebra II

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Lecture 15

Let us discuss solutions to selected homework questions.

1. (HW1) For each of the following matrices A, viewed as a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$,

- compute the rank of A;
- describe all eigenvalues and eigenvectors of A;
- determine whether there exists a change of coordinates making the matrix A diagonal.

(**a**) $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$; (**b**) $A = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 4 \\ 1 & -1 & 4 \end{pmatrix}$; (**c**) $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & 5 & 2 \end{pmatrix}$.

Solution: (a) $\text{rk}(A) = 1$ (all columns are the same, so there is just one linearly independent column), eigenvectors are 0 and 3, there are two linearly independent eigenvectors $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ for the first of them (and every eigenvector is their linear combination), and every eigenvector for the second of them is proportional to $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Since there are three linearly independent eigenvectors, the matrix is similar to a diagonal matrix.

(b) $\text{rk}(A) = 3$, eigenvectors are 2 and 3, every eigenvector for the first one is proportional to $\begin{pmatrix} -4 \\ -2 \\ 1 \end{pmatrix}$, every eigenvector for the second one is proportional to $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. Since we do not have three linearly independent eigenvectors, the matrix is not similar to a diagonal matrix.

(c) $\mathrm{rk}(A) = 3$, eigenvectors are -2, 1, and 3, every eigenvector for the first one is proportional to $\begin{pmatrix} 1/4 \\ -1/2 \\ 1 \end{pmatrix}$, every eigenvector for the second one is proportional to $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, every eigenvector for the third one is proportional to $\begin{pmatrix} 1/9 \\ 1/3 \\ 1 \end{pmatrix}$.

2. (HW4) For the subspace $U \in \mathbb{R}^5$ spanned by the vectors $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$, determine some basis for the subspace $U^\perp$. The scalar product on $\mathbb{R}^5$ is the standard one $(v, w) = v_1 w_1 + \ldots + v_5 w_5$.

Solution: The orthogonal complement of our subspace consists of all vectors which are orthogonal to both of the spanning vectors, that is vectors $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$ for which $x_1 + x_2 + x_3 + x_4 = 0$ and $x_1 - x_3 + x_5 = 0$. Solving this system, we observe that the variables x_3, x_4, x_5 are free, and we get a parametrisation of the orthogonal complement: $\begin{pmatrix} u - w \\ w - 2u - v \\ u \\ v \\ w \end{pmatrix}$, where $u, v, w \in \mathbb{R}$.

3. (HW5) Use the Sylvester's criterion to find all values of the parameter a for which the quadratic form $2x_1^2 + x_2^2 + x_3^2 + 2ax_1x_2 + 2x_1x_3 + (2 - 2a)x_2x_3$ on $\mathbb{R}^3$ is positive definite.

Solution: The matrix of the corresponding bilinear form is

$$A = \begin{pmatrix} 2 & a & 1 \\ a & 1 & 1-a \\ 1 & 1-a & 1 \end{pmatrix}.$$

We have $\Delta_1 = 2$, $\Delta_2 = 2 - a^2$, $\Delta_3 = -5a^2 + 6a - 1$. All these numbers are positive if and only if $|a| < \sqrt{2}$ and $1/5 < a < 1$ (since the roots of $-5a^2 + 6a - 1$ are $1/5$ and 1). In fact, the second condition implies the first one, so we get the answer $1/5 < a < 1$.

4. (HW6) Find an orthonormal basis of eigenvectors for the matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

For the quadratic form $q(x) = (Ax, x)$, compute the maximal and the minimal value of $q(x)$ on the unit sphere $S = \{x \mid (x, x) = 1\}$.

Solution: Eigenvalues are 0, 1, and 3, so the corresponding eigenvectors are automatically orthogonal. Normalizing them (dividing by lengths), we get an orthonormal basis of eigenvectors

$$\frac{1}{\sqrt{3}}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}.$$

We have

$$\max_{(x,x)=1} q(x) = 3, \qquad \min_{(x,x)=1} q(x) = 0,$$

as these values are given by the max/min eigenvalue of our matrix. The minimum is reached on the first vector from the basis of eigenvectors, the maximum — on the third vector.