1212: Linear Algebra II

Dr. Vladimir Dotsenko (Vlad)

Lecture 16

Example 1. Let $V = \mathbb{C}^n$ with the *standard scalar product*

$$\left(\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}\right) = x_1\overline{y}_1 + x_2\overline{y}_2 + \cdots + x_n\overline{y}_n.$$

All the three properties from the definition of a Hermitian space are trivially true.

Lemma 1. *For every Hermitian scalar product and every basis* $e_1, \ldots, e_n$ *of* V*, we have*

$$(x_1e_1 + \ldots + x_ne_n, y_1e_1 + \ldots + y_ne_n) = \sum_{i,j=1}^{n} a_{ij}x_i\overline{y}_j,$$

where $a_{ij} = (e_i, e_j)$.

This follows immediately from the sesquilinearity property of Hermitian scalar products.

Similarly to how one defines orthogonal and orthonormal systems in Euclidean spaces, we say that a system of vectors $e_1, \ldots, e_k$ of a Hermitian space V is orthogonal, if it consists of nonzero vectors, which are pairwise orthogonal: $(e_i, e_j) = 0$ for $i \neq j$. An orthogonal system is said to be orthonormal, if $(e_i, e_i) = 1$ for all i. There is an obvious version of Gram–Schmidt orthogonalisation procedure for Hermitian scalar products; it ensures that every Hermitian vector space has an orthonormal basis.

Note that a basis $e_1, \ldots, e_n$ of V is orthonormal if and only if

$$(x_1e_1 + \ldots + x_ne_n, y_1e_1 + \ldots + y_ne_n) = x_1\overline{y}_1 + \ldots + x_n\overline{y}_n.$$

Definition 1. Let φ be a linear transformation of a Hermitian vector space V. The *adjoint* linear transformation $\varphi^\dagger$ is uniquely determined by the formula

$$(\varphi v_1, v_2) = (v_1, \varphi^\dagger v_2)$$

that should be valid for all v_1 and v_2. (Indeed, substituting elements from an orthonormal basis for v_1 recovers all coordinates of $\varphi^\dagger v_2$).

Remark 1. Note that $(\varphi^\dagger)^\dagger = \varphi$. Indeed, we have

$$(v_1, (\varphi^\dagger)^\dagger(v_2)) = (\varphi^\dagger(v_1), v_2) = \overline{(v_2, \varphi^\dagger(v_1))} = \overline{(\varphi(v_2), v_1)} = (v_1, \varphi(v_2)),$$

from which it follows that

$$(\varphi^\dagger)^\dagger(v_2) = \varphi(v_2),$$

and it implies

$$(\varphi^\dagger)^\dagger = \varphi.$$

Proposition 1. *Let* $e_1, \dots, e_n$ *be an orthonormal basis of a Hermitian vector space* V*, and let the linear transformation* φ *be represented by a matrix* B *relative to this basis. Then the linear transformation* $\varphi^\dagger$ *is represented by the matrix* $\overline{B^T}$*.*

Definition 2. A linear transformation φ of a Hermitian vector space is said to be symmetric if $(\varphi(v_1), v_2) = (v_1, \varphi(v_2))$ for all v_1, v_2, in other words, if $\varphi^\dagger = \varphi$. A linear transformation is said to be unitary, if $(\varphi(v_1), \varphi(v_2)) = (v_1, v_2)$, in other words, if $\varphi^\dagger = \varphi^{-1}$. A linear transformation φ is said to be normal if $\varphi\varphi^\dagger = \varphi^\dagger\varphi$.

Theorem 1. *A normal linear transformation admits an orthonormal basis of eigenvectors.*

Proof. We know that two commuting linear transformations have a common eigenvector, so there exists a vector v for which $\varphi(v) = cv$, and $\varphi^\dagger(v) = c'v$. We note that

$$c(v, v) = (cv, v) = (\varphi(v), v) = (v, \varphi^\dagger(v)) = (v, c'v) = \overline{c'}(v, v),$$

so $c = \overline{c'}$, although this will not be crucial for us. What is crucial is that for Hermitian scalar products one can define orthogonal complements etc., and it is easy to see that the orthogonal complement of v is an invariant subspace of both φ and $\varphi^\dagger$: if $w \in \operatorname{span}(v)^\perp$, then

$$(\varphi(w), v) = (w, \varphi^\dagger(v)) = (w, c'v) = \overline{c'}(w, v) = 0,$$
$$(\varphi^\dagger(w), v) = (w, (\varphi^\dagger)^\dagger(v)) = (w, \varphi(v)) = (w, cv) = c(w, v) = 0.$$

This allows us to proceed by induction on dimension of V: we found an eigenvector, and by the induction hypothesis we have a basis of eigenvectors in the orthogonal complement of that eigenvector. □

Theorem 2. *Eigenvalues of a symmetric linear transformation are real. Every Hermitian linear transformation admits a orthonormal basis of eigenvectors.*

Proof. Let v be an eigenvector of the given Hermitian linear transformation φ. Then the eigenvalues of φ are, as we proved earlier, self-conjugate, and therefore real. The rest follows from the fact that a symmetric linear transformation is normal. □

Remark 2. This theorem can be used to deduce the theorem on symmetric matrices we proved before, but the proof will invoke complex numbers. The proof we discussed before is a bit more consistent with the context of that theorem.