1212: Linear Algebra II

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Lecture 2

Rank and nullity of a linear map

Last time we defined the notions of the kernel and the image of a linear map $\varphi\colon V \to W$.

Definition 1. The rank of a linear map φ, denoted by $\mathrm{rk}(\varphi)$, is the dimension of the image of φ. The nullity of φ, denoted by $\mathrm{null}(\varphi)$, is the dimension of the kernel of φ.

Example 1. Let $\iota\colon V \to V$ be the identity map. Then $\mathrm{null}(\iota) = 0$, and $\mathrm{rk}(\iota) = \dim(V)$.

Example 2. Let $0\colon V \to W$ be the map sending every vector v to $0 \in W$. Then $\mathrm{null}(\iota) = \dim(V)$, and $\mathrm{rk}(\iota) = 0$.

Example 3. Let $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^3$ be linear map given by the left multiplication by the matrix $A = \begin{pmatrix} 2 & 1 \\ 4 & 2 \\ 10 & 5 \end{pmatrix}$. Then $A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x+y \\ 4x+2y \\ 10x+5y \end{pmatrix} = (2x+y)\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$, so

- every vector in $\mathrm{Im}(\varphi)$ is proportional to $\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$, therefore $\mathrm{rk}(\varphi) = 1$,
- $\varphi(\begin{pmatrix} x \\ y \end{pmatrix}) = 0$ implies $2x + y = 0$, so $y = -2x$, and $\begin{pmatrix} x \\ y \end{pmatrix} = x\begin{pmatrix} 1 \\ -2 \end{pmatrix}$, therefore $\mathrm{null}(\varphi) = 1$.

In fact, it is always enough to compute just one of those two numbers to know the other, as the following result shows.

Theorem 1 (Rank-nullity theorem). *For a linear map $\varphi\colon V \to W$, we have*

$$\mathrm{rk}(\varphi) + \mathrm{null}(\varphi) = \dim(V).$$

Proof. Let us fix some coordinate systems $e_1, \ldots, e_n \in V$ and $f_1, \ldots, f_m \in W$, and represent φ by the matrix $A = A_{\varphi,\mathbf{e},\mathbf{f}}$.

Since $\mathrm{Ker}(\varphi)$ consists of vectors v such that $\varphi(v) = 0$, we see that $\mathrm{null}(\varphi)$ is the dimension of the solution space of the system $Ax = 0$, where $x = v_{\mathbf{e}}$. This dimension is equal to the number of free variables, that is the number of non-pivotal columns of the reduced row echelon form.

Also, $\mathrm{Im}(\varphi)$ consists of all vectors of the form $\varphi(v)$ with $v \in V$. Since each v can be written as $v = x_1e_1 + \cdots + x_ne_n$, we can write $\varphi(v) = x_1\varphi(e_1) + \cdots + x_n\varphi(e_n)$, so $\mathrm{Im}(\varphi)$ is spanned by $\varphi(e_1)$, …, $\varphi(e_n)$. Columns of coordinates of these vectors are precisely the columns of the matrix A, by its definition. In fact, we may assume A to be in its reduced row echelon form, since bringing it to that form can be done by multiplying A by an invertible matrix on the left (the product of elementary matrices corresponding to row operations bringing A to reduced row echelon form), and this, as we know, merely corresponds to a change of basis in W. If A is a reduced row echelon form matrix, its columns with pivots are some of the standard unit vectors, and all other columns are their combinations. Thus $\mathrm{rk}(\varphi)$ is the number of pivotal columns of the reduced row echelon form.

Altogether, $\mathrm{rk}(\varphi) + \mathrm{null}(\varphi)$ is the total number of columns, which is $\dim(V)$. □

Remark 1. Note that in fact we can change bases in both the source and the target of φ, and hence do both elementary row and elementary column operations without changing the rank and the nullity. This is useful for some computations. By doing both elementary row and elementary column operations, every $m \times n$-matrix A can be brought to the form $\begin{pmatrix} I_k & 0_{(n-k)\times k} \\ 0_{k\times(m-k)} & 0_{(n-k)\times(m-k)} \end{pmatrix}$, where $0_{a\times b}$ is the $a \times b$-matrix whose all entries are equal to 0. Here k is the rank of the corresponding linear map.

The following simple fact concerning subspaces may have eluded us so far.

Lemma 1. *Suppose that U is a subspace of a vector space V. Then any basis of U can be extended to a basis of V, in particular, $\dim(U) \leqslant \dim(V)$.*

Proof. If $U = V$, there is nothing to prove. Otherwise, let us take a basis $u_1, \ldots, u_k$ of U. There exists a vector $v \in V$ which cannot be represented as a linear combination of $u_1, \ldots, u_k$. Therefore, $u_1, \ldots, u_k, v$ are linearly independent. Now we replace U by $\mathrm{span}(u_1, \ldots, u_k, v)$. If that subspace is equal to V, we are done. Otherwise, we continue in the same manner. Every time dimension increases by 1, so it cannot continue infinitely long, since the number of elements in a linearly independent system cannot exceed $\dim(V)$. Therefore, at some stage we obtain the whole of V, and the statement follows. □

Another proof of rank-nullity theorem. This other proof of the rank-nullity theorem will be useful in some subsequent classes.

The kernel of φ is a subspace of V. Let us choose a basis $f_1, \ldots, f_k$ of $\mathrm{Ker}(\varphi)$. We can extend this basis to a basis of V by adjoining vectors $g_1, \ldots, g_l$. I claim that the vectors $\varphi(g_1), \ldots, \varphi(g_l)$ form a basis of $\mathrm{Im}(\varphi)$. Since

$$\varphi(a_1 f_1 + \cdots + a_k f_k + b_1 g_1 + b_l g_l) = a_1 \varphi(f_1) + \cdots + a_k \varphi(f_k) + b_1 \varphi(g_1) + \cdots + b_l \varphi(g_l),$$

and $\varphi(f_i) = 0$, the vectors $\varphi(g_1), \ldots, \varphi(g_l)$ form a spanning set. Suppose they are linearly dependent, so

$$0 = c_1 \varphi(g_1) + \cdots + c_l \varphi(g_l) = \varphi(c_1 g_1 + \cdots + c_l g_l).$$

But this would imply $c_1 g_1 + \cdots + c_l g_l \in \mathrm{Ker}(\varphi)$, so

$$c_1 g_1 + \cdots + c_l g_l = d_1 f_1 + \cdots + d_k f_k,$$

contradicting the basis property of f_i and g_j taken together. Therefore, $\mathrm{rk}(\varphi) + \mathrm{null}(\varphi) = l + k = \dim(V)$.