Midterm test, March 4, 2015

ATTEMPT ALL QUESTIONS

Please put your name and student number on each of the sheets you are handing in.

You are allowed to use theoretical results from class if you state clearly the result you are using. (E.g., "in class, it was proved that a symmetric matrix with real entries has an orthonormal basis of eigenvectors").

The number of marks you may get for a full solution to a question is written next to it. You should provide complete solution (both the answer and a derivation of the answer) to get full marks. Non-programmable calculators are allowed. Use of mobile phones is **strictly** prohibited.

1. (**a**) (5 points) Write down definitions of a linear map, a linear transformation, and an invariant subspace of a linear transformation.

(**b**) (15 points) Is the subspace U of $\mathbb{R}^4$ spanned by $\begin{pmatrix} 1 \\ 1 \\ 4 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ -1 \\ -1 \\ 1 \end{pmatrix}$ an invariant subspace of the linear transformation φ whose matrix relative to the basis of standard unit vectors is

$$\begin{pmatrix} 0 & 3 & -3 & -1 \\ 1 & 3 & -1 & 0 \\ 7 & 12 & 2 & 3 \\ -3 & -6 & 0 & -1 \end{pmatrix}?$$

Explain your answer.

2. (**a**) (5 points) Write down the definition of a bilinear form on a real vector space. Which symmetric bilinear forms are said to be positive definite?

(**b**) (15 points) A quadratic form Q on the three-dimensional space with a basis e_1, e_2, e_3 is defined by the formula

$$Q(xe_1 + ye_2 + ze_3) = 3x^2 + 2axy + (2 - 2a)xz + (a + 2)y^2 + 2ayz + 3z^2$$

Find all values of the parameter a for which $Q(v) > 0$ for all $v \neq 0$.

3. (**a**) (5 points) Define a Euclidean vector space. Which bases of a Euclidean space V are called orthogonal? orthonormal?

(**b**) (5 points) Show that the $f_1 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$, $f_2 = \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix}$, and $f_3 = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$ form a basis of $\mathbb{R}^3$.

(**c**) (10 points) Find the orthogonal basis of $\mathbb{R}^3$ which is the output of the Gram-Schmidt orthogonalisation applied to the basis from the previous question. (The scalar product on $\mathbb{R}^3$ is the standard one).