MA 1212: Linear Algebra II
Tutorial problems, January 22, 2015

**1.** (**a**) The reduced column echelon form of the matrix whose columns are the given vectors is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4/7 & 1/7 & -1/7 \end{pmatrix}$$

so the columns of this matrix are linearly independent, and either the original vectors or the columns of the reduced column echelon form can be taken for a basis.

(**b**) The reduced column echelon form of the matrix whose columns are the given vectors is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -7/5 & -6/5 & 13/5 & 0 \end{pmatrix},$$

so the dimension of the span of the column space of this matrix is 3, and the nonzero columns of the reduced column echelon form can be taken for a basis.

**2.** The intersection is described by the system of equations

$$c_1e_1 + c_2e_2 + c_3e_3 - c_4f_1 - c_5f_2 - c_6f_3 = 0,$$

where $e_1, e_2, e_3$ are columns of the reduced column echelon form for the first matrix, $f_1, f_2, f_3, f_4$ are the nonzero columns of the reduced column echelon form for the second matrix. The matrix of this system is

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 4/7 & 1/7 & -1/7 & 7/5 & 6/5 & -13/5 \end{pmatrix}$$

and it reduced row echelon form is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 47/69 & -32/23 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 47/69 & -32/23 \end{pmatrix}$$

so $c_5$ and $c_6$ are free variables. Setting $c_5 = 1$, $c_6 = 0$, we obtain $c_4 = -47/69$; setting $c_5 = 0$, $c_6 = 1$, we obtain $c_4 = 32/23$. The corresponding basis vectors $c_4 f_1 + c_5 f_2 + c_6 f_3$ are, respectively, $\begin{pmatrix} -47/69 \\ 1 \\ 0 \\ -17/69 \end{pmatrix}$ and $\begin{pmatrix} 32/23 \\ 0 \\ 1 \\ 15/23 \end{pmatrix}$.

**3.** For $U = \text{span}(v_1, v_2)$ to be invariant, it is necessary and sufficient to have $\varphi(v_1), \varphi(v_2) \in U$. Indeed, this condition is necessary because we must have $\varphi(U) \subset U$, and it is sufficient because each vector of $U$ is a linear combination of $v_1$ and $v_2$.

We have $\varphi(v_1) = Av_1 = \begin{pmatrix} -3 \\ 8 \\ -8 \end{pmatrix}$ and $\varphi(v_2) = Av_2 = \begin{pmatrix} -1 \\ 8 \\ -8 \end{pmatrix}$. It just remains to see if there are scalars $x, y$ such that $\varphi(v_1) = xv_1 + yv_2$ and scalars $z, t$ such that $\varphi(v_2) = zv_1 + tv_2$. Solving the corresponding systems of linear equations, we see that there are solutions: $\varphi(v_1) = -3v_1 + 5v_2$ and $\varphi(v_2) = -v_1 + 7v_2$. Therefore, this subspace is invariant.