MA 1111: Linear Algebra I
Selected answers/solutions to the assignment due October 22, 2015

1. (**a**) $A + B$ and BA are not defined, $AB = \begin{pmatrix} 8 \\ 5 \end{pmatrix}$; (**b**) $A + B$ and BA are not defined, $AB = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$;

(**c**) $A + B$ is not defined, $BA = \begin{pmatrix} 9 & 1 & 9 \\ 1 & 0 & 2 \\ 15 & 2 & 12 \end{pmatrix}$, $AB = \begin{pmatrix} 7 & 26 \\ 5 & 14 \end{pmatrix}$;

(**d**) $A + B = \begin{pmatrix} 4 & 8 \\ 2 & 3 \end{pmatrix}$, $BA = \begin{pmatrix} 8 & 16 \\ 3 & 6 \end{pmatrix}$, $AB = \begin{pmatrix} 8 & 12 \\ 4 & 6 \end{pmatrix}$.

2. The easiest thing to do is to apply the algorithm from the lecture: take the matrix $(A \mid I_n)$ and bring it to the reduced row echelon form; the result is $(I_n \mid A^{-1})$ if the matrix is invertible, and has $(R \mid B)$ with $R \neq I_n$ otherwise.

(**a**) $\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}$ is invertible, the inverse is $\begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix}$;

(**b**) $\begin{pmatrix} 6 & 4 \\ 3 & 2 \end{pmatrix}$ is not invertible, since the reduced row echelon form of $(A \mid I)$ is $\begin{pmatrix} 1 & 2/3 & 0 & 1/2 \\ 0 & 0 & 1 & -2 \end{pmatrix}$ so the matrix on the left is not the identity;

(**c**) $\begin{pmatrix} 1 & 1 & 3 \\ 1 & 2 & 0 \end{pmatrix}$ is not invertible, since in class we proved that only square matrices are invertible;

(**d**) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$ is invertible; the inverse is $\begin{pmatrix} 3 & -3 & 1 \\ -5/2 & 4 & -3/2 \\ 1/2 & -1 & 1/2 \end{pmatrix}$.

3. (**a**) Suppose that A is a $k \times l$-matrix, and B is an $m \times n$-matrix. In order for AB to be defined, we must have $l = m$. In order for BA to be defined, we must have $n = k$. Consequently, the size of matrix AB is $k \times n = n \times n$, and the size of the matrix BA is $m \times l = m \times m$, which is exactly what we want to prove.

(**b**) We have

$$\begin{aligned} \mathrm{tr}(UV) = (UV)_{11} + (UV)_{22} + \ldots + (UV)_{nn} = (U_{11}V_{11} + U_{12}V_{21} + \ldots + U_{1n}V_{n1}) + \\ (U_{21}V_{12} + U_{22}V_{22} + \ldots + U_{2n}V_{n2}) + \ldots + (U_{n1}V_{1n} + U_{n2}V_{2n} + \ldots + U_{nn}V_{nn}), \end{aligned}$$

and

$$\begin{aligned} \mathrm{tr}(VU) = (VU)_{11} + (VU)_{22} + \ldots + (VU)_{nn} = (V_{11}U_{11} + V_{12}U_{21} + \ldots + V_{1n}U_{n1}) + \\ (V_{21}U_{12} + V_{22}U_{22} + \ldots + V_{2n}U_{n2}) + \ldots + (V_{n1}U_{1n} + V_{n2}U_{2n} + \ldots + V_{nn}U_{nn}), \end{aligned}$$

so both traces are actually equal to the sum of all products $U_{ij}V_{ji}$, where i and j range from 1 to n. For the example $U = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $V = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ from class, we have $UV = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $VU = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, so even though these matrices are not equal, their traces are equal.

4. (**a**) We have $A^2 = \begin{pmatrix} a^2+bc & b(a+d) \\ c(a+d) & d^2+bc \end{pmatrix}$. (**b**) Since $A^2 = I_2$, from the previous formula we have $b(a+d) = c(a+d) = 0$. If $a+d=0$, we have $\mathrm{tr}(A) = 0$, and everything is proved. Otherwise, if $a+d \neq 0$, we have $b = c = 0$, so $a^2 = 1 = d^2$, and either $a = d = 1$ or $a = d = -1$ or $a = 1, d = -1$ or $a = -1, d = 1$. In the first case $A = I$, in the second case $A = -I$, in the remaining two cases $\mathrm{tr}(A) = a + d = 0$ (which is contradiction since in this case we assumed $a + d \neq 0$). (**c**) For example, the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ works.

5. For example, $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ would work.