1111: Linear Algebra I

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Lecture 16

Given a field F, one can consider vector spaces over F, that is vector spaces where elements of F play the role of scalars. The flexibility of choosing scalars for the vector space can sometimes be very useful.

Coin weighing problem

Let us look at the following question.

> Given 101 coins of various shapes and denominations, one knows that if you remove any one coin, the remaining 100 coins can be divided into two groups of 50 of equal total weight. Show that all the coins are of the same weight.

Let us suppose that there exist examples where weights are not all equal to each other.

We proceed in several steps.

First, we consider the case when all weights $x_1, \ldots, x_{101}$ of all coins in a counterexample are non-negative integers.

Lemma 1. *The weights of the coins are either all even or all odd.*

Proof. Denote $S = x_1 + \cdots + x_{101}$. Then $S - x_i$ is divisible by 2 for all i, because we can split all coins except for the coin number i into two groups of equal total weight, so $S - x_i$ is twice that weight. Therefore, $x_i - x_j = (S - x_j) - (S - x_i)$ is divisible by 2 also. □

Among all examples where weights which are not all equal to each other, let us choose the example with the least possible total weight. If all the weights are even, we can divide them by 2, and get a set of coins satisfying our assumption of smaller total weight. If all the weights are odd, we can subtract 1 from each, and get a set of coins satisfying our assumptions of smaller total weight. In either case we get a contradiction with the minimality of the total weight in our set. Thus, in this case there may be no counterexamples.

Next, we handle the examples where all weights are integers, possibly negative, even though from the practical point of view this is absurd. Basically, given such a set, we can add the same large integer to all weights, obtaining a set where all weights are positive integers; this is a situation we already handled.

Second, we suppose all weights are rational. Then, multiplying by common denominator, we get a set of coins satisfying our assumptions where all weights are integers; this is a situation we already handled.

Finally, suppose weights are arbitrary real numbers. Note that the conditions we impose can be expressed as a system of linear equations with rational coefficients! Saying that there is a solution where not all weights are equal is essentially saying that if we let $x_1 = 1$, there is a solution where not all coordinates are equal to 1, so this system of equations has at least 2 solutions. But this is a property that "does not depend on scalars", — whether we view our system of equations as a system with rational coefficients or with real coefficients, we do the same, compute the reduced row echelon form. If there is a solution different from the solution $x_1 = x_2 = \cdots = x_{101} = 1$ over real numbers, there must be free unknowns! Setting all these free unknowns equal to zero, we shall obtain a solution with rational coordinates where not all coordinates are equal. But we already proved that the latter was impossible.

Linear independence, span, basis

By definition of a vector space, we can form arbitary linear combinations: if $v_1, \dots, v_k$ are vectors and $c_1, \dots, c_k$ are scalars, then $c_1 v_1 + \dots + c_k v_k$ is a vector which is called the linear combination of $v_1, \dots, v_k$ with coefficients $c_1, \dots, c_k$.

All the definitions that we gave in the case of $\mathbb{R}^n$ proceed in the same way. Below we assume that V is a vector space over real numbers (but one can use any other field if necessary).

Definition 1. A system of vectors $v_1, \dots, v_k \in V$ is said to be linearly independent if the only linear combination of these vectors that is equal to zero is the combination where all the coefficients are equal to zero.

Note that the property stating that if $c \cdot v = 0$ then $c = 0$ or $v = 0$ can be rephrased as follows: one non-zero vector is always linearly independent.

Definition 2. A system of vectors $v_1, \dots, v_k \in V$ is said to be *a spanning set* if every vector in V can be represented as their linear combination.

In case the given vectors do not form a spanning set, they span a subspace of V. Let us make this precise.

Definition 3. The linear span of vectors $v_1, \dots, v_k \in V$ is the set of all linear combinations of these vectors,

$$\mathrm{span}(v_1, \dots, v_k) = \{c_1 v_1 + \dots + c_k v_k : c_1, \dots, c_k \in \mathbb{R}\} .$$

The following statement is easy to check, and is left as an exercise.

Lemma 2. *For any vectors* $v_1, \dots, v_k$*, the set* $\mathrm{span}(v_1, \dots, v_k)$ *is a subspace of* V*.*

Definition 4. A system of vectors $v_1, \dots, v_k \in V$ is said to form a *basis* of V, if it is linearly independent and spans V.