

MA 1111/1212: Linear Algebra  
Tutorial problems, October 21, 2015

1. Let us write down the corresponding one-row representatives of these permutations (for which we permute the columns to create the natural order in the top row): the matrix  $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 5 & 4 & 3 \end{pmatrix}$  corresponds to 2, 1, 5, 4, 3, the matrix  $\begin{pmatrix} 1 & 4 & 2 & 3 & 5 \\ 2 & 1 & 5 & 3 & 4 \end{pmatrix}$  corresponds to 2, 5, 3, 1, 4, and the matrix  $\begin{pmatrix} 5 & 3 & 1 & 4 & 2 \\ 3 & 5 & 2 & 4 & 1 \end{pmatrix}$  corresponds to 2, 1, 5, 4, 3. Therefore the first and the third matrix do represent the same permutation. The permutation 2, 1, 5, 4, 3 is even (it has 4 inversions), and the permutation 2, 5, 3, 1, 4 is odd (it has 3 inversions).

2. Clearly, we must have  $i = 2$  (to have all the numbers present in the top row) and  $\{j, k\} = \{3, 4\}$ . For the choice  $j = 3$ ,  $k = 4$ , the permutation is even (since there are 8 inversions in total in the two rows), so for the other choice the permutation is odd.  
*Answer:*  $i = 2$ ,  $j = 4$ ,  $k = 3$ .

3. The last two rows are proportional, so subtracting from the third row three times the second row, we obtain a matrix with a zero row whose determinant is zero.

4. (a) If we subtract from the second row twice the first row, and add to the third row the first row, we get the matrix  $\begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 6 & 3 \end{pmatrix}$ ; now, we have

$$\det \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 6 & 3 \end{pmatrix} = -3 \det \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix} = -3 \det \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{pmatrix} = 9 \det \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} = 9.$$

(b) We have

$$\begin{aligned} \det \begin{pmatrix} 2 & 1 & -3 & 0 \\ 1 & 5 & 2 & -1 \\ 5 & 0 & 13 & 8 \\ 0 & 1 & 2 & 1 \end{pmatrix} &= -\det \begin{pmatrix} 1 & 5 & 2 & -1 \\ 2 & 1 & -3 & 0 \\ 5 & 0 & 13 & 8 \\ 0 & 1 & 2 & 1 \end{pmatrix} = -\det \begin{pmatrix} 1 & 5 & 2 & -1 \\ 0 & -9 & -7 & 2 \\ 0 & -25 & 3 & 13 \\ 0 & 1 & 2 & 1 \end{pmatrix} = \\ \det \begin{pmatrix} 1 & 5 & 2 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & -25 & 3 & 13 \\ 0 & -9 & -7 & 2 \end{pmatrix} &= \det \begin{pmatrix} 1 & 5 & 2 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 53 & 38 \\ 0 & 0 & 11 & 11 \end{pmatrix} = 11 \det \begin{pmatrix} 1 & 5 & 2 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 53 & 38 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \\ -11 \det \begin{pmatrix} 1 & 5 & 2 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 53 & 38 \end{pmatrix} &= -11 \det \begin{pmatrix} 1 & 5 & 2 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -15 \end{pmatrix} = 11 \cdot 15 \det \begin{pmatrix} 1 & 5 & 2 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = 165. \end{aligned}$$

**Optional question:** Subtracting, for each  $i = 1, \dots, n - 1$ , the row  $i$  from the row  $i + 1$ , we obtain the matrix whose diagonal entry are all equal to 1, and all entries below diagonal are equal to zero; its determinant is equal to 1.