MA 1111/1212: Linear Algebra
Tutorial problems, November 4, 2015

These solutions show a range of ideas one can utilize, from applying the definition directly to using various results on matrices that we proved in class. In fact, there are many different ways to approach each of these problems, and one may choose either of the ways, as long as the solution is complete and coherent.

1. These vectors are proportional, $v_2 = -7v_1$, so they certainly are linearly dependent. They do not span $\mathbb{R}^2$, since any linear combination of them will be proportional to each of them as well. Consequently, they do not form a basis of $\mathbb{R}^2$.

2. The matrix $A = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$ formed by these vectors is invertible (its determinant is equal to -2), so the system $Ax = 0$ has only the trivial solution (and hence the vectors are linearly independent), and the system $Ax = b$ has solutions for every b (and hence the vectors span $\mathbb{R}^2$). Consequently, these vectors form a basis of $\mathbb{R}^2$.

3. The reduced row echelon form of the matrix $\begin{pmatrix} -1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ formed by these vectors is $\begin{pmatrix} 1 & 0 & 1/3 \\ 0 & 1 & 2/3 \end{pmatrix}$. Each row of this reduced row echelon form has a pivot so the given vectors span $\mathbb{R}^2$; the last column has no pivot, so these vectors are linearly dependent. Therefore, they do not form a basis of $\mathbb{R}^2$.

4. By inspection, the sum of these three vectors is equal to zero, so they are not linearly independent. In other words, if we denote by A the matrix formed by these vectors, the system $Ax = 0$ has a nontrivial solution. By Fredholm's alternative (here it is important that A is a square matrix), the system $Ax = b$ has no solutions for some b, so these vectors do not span $\mathbb{R}^3$. Clearly, they do not form a basis of $\mathbb{R}^3$ either.

5. The reduced row echelon form of the matrix $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ formed by these vectors is I_3, the identity matrix (this can be seen in several ways, e.g. by computing it directly, or by computing the determinant, which is equal to $2 \neq 0$). Each row and each column of the identity matrix has a pivot, so the given vectors are linearly independent, span $\mathbb{R}^3$, and form a basis of $\mathbb{R}^3$.