1111: Linear Algebra I

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Lecture 11

Previously on...

For an $n \times n$-matrix A with entries a_{ij}, we denote by A^{ij} the i, j-*minor* of A, that is the determinant of the $(n-1) \times (n-1)$-matrix obtained from A by removing the i-th row and the j-th column. For example, let us consider the matrix $A = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 1 & -2 \\ 0 & 1 & 1 \end{pmatrix}$. Then $A^{11} = 3$, $A^{12} = 2$, $A^{13} = 2$, $A^{21} = 3$, $A^{22} = 1$, $A^{23} = 1$, $A^{31} = -6$, $A^{32} = -2$, $A^{33} = -5$.

Cofactors are "minors with signs": for the given matrix A, its cofactors C^{ij} are defined by the formula $C^{ij} = (-1)^{i+j} A^{ij}$.

In simple words, the extra signs follow a chessboard pattern: we put $+1$ in the top left corner, and then keep changing signs once we cross borders between rows / columns.

For example, for the matrix $A = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 1 & -2 \\ 0 & 1 & 1 \end{pmatrix}$, we have $C^{11} = 3$, $C^{12} = -2$, $C^{13} = 2$, $C^{21} = -3$, $C^{22} = 1$, $C^{23} = -1$, $C^{31} = -6$, $C^{32} = 2$, $C^{33} = -5$.

Row expansion of the determinant

Our next goal is to prove the following formula:

$$\det(A) = a_{i1}C^{i1} + a_{i2}C^{i2} + \cdots + a_{in}C^{in},$$

in other words, if we multiply each element of the i-th row of A by its cofactor and add results, the number we obtain is equal to $\det(A)$.

Let us first handle the case $i = 1$. In this case, once we write the corresponding minors with signs, the formula reads

$$\det(A) = a_{11}A^{11} - a_{12}A^{12} + \cdots + (-1)^{n+1}a_{1n}A^{1n}.$$

Let us examine the formula for $\det(A)$. It is a sum of terms corresponding to pick n elements representing each row and each column of A. In particular, it involves picking an element from the first row. This can be one of the elements $a_{11}, a_{12}, \dots, a_{1n}$. What remains, after we picked the element a_{1k}, is to pick other elements; now we have to avoid the row 1 and the column k. This means that the elements we need to pick are precisely those involved in the determinant A^{1k}, and we just need to check that the signs match.

Row expansion of the determinant

In fact, it is quite easy to keep track of all signs. The column $\begin{pmatrix} 1 \\ k \end{pmatrix}$ in the two-row notation does not add inversions in the first row, and adds $k-1$ inversions in the second row, since k appears before $1, 2, \ldots, k-1$. This shows that the signs indeed have that mismatch $(-1)^{k-1} = (-1)^{k+1}$, as we claim.

Note that the case of arbitrary i is not difficult. We can just reduce it to the case of $i = 1$ by performing $i-1$ row swaps. This multiplies the determinant by $(-1)^{i-1}$, which matches precisely the signs in cofactors in the formula

$$\det(A) = a_{i1}C^{i1} + a_{i2}C^{i2} + \cdots + a_{in}C^{in} \ .$$

“Wrong row expansion”

In fact, another similar formula also holds: for $i \neq j$, we have

$$a_{i1}C^{j1} + a_{i2}C^{j2} + \cdots + a_{in}C^{jn} = 0 \ .$$

It follows instantly from what we already proved: take the matrix A' which is obtained from A by replacing the j-th row by a copy of the i-th row. Then the left hand side is just the j-th row expansion of $\det(A')$, and it remains to notice that $\det(A') = 0$ because this matrix has two equal rows.

These results altogether can be written like this:

$$a_{i1}C^{j1} + a_{i2}C^{j2} + \cdots + a_{in}C^{jn} = \det(A)\delta_i^j \ .$$

Here δ_i^j is the *Kronecker symbol*; it is equal to 1 for $i = j$ and is equal to zero otherwise. In matrix notation, $A \cdot C^T = \det(A) \cdot I_n$. (Note that here we have a product of matrices in the first case, and the product of the matrix I_n and the scalar $\det(A)$ in the second case).

Adjugate matrix

The transpose of cofactor matrix $C = (C^{ij})$ is called the *adjugate matrix* of the matrix A, and is denoted $\mathrm{adj}(A)$. (Historically it was called the adjoint matrix, but now that term is used for something else). For example, for the matrix $A = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 1 & -2 \\ 0 & 1 & 1 \end{pmatrix}$ we have $C = \begin{pmatrix} 3 & -2 & 2 \\ -3 & 1 & -1 \\ -6 & 2 & -5 \end{pmatrix}$, and $\mathrm{adj}(A) = \begin{pmatrix} 3 & -3 & -6 \\ -2 & 1 & 2 \\ 2 & -1 & -5 \end{pmatrix}$.

We already proved one half of the following result:

Theorem. For each $n \times n$-matrix A, we have

$$A \cdot \mathrm{adj}(A) = \mathrm{adj}(A) \cdot A = \det(A) \cdot I_n.$$

Adjugate matrix

Theorem. For each $n \times n$-matrix A, we have

$$A \cdot \mathrm{adj}(A) = \mathrm{adj}(A) \cdot A = \det(A) \cdot I_n.$$

The other half is proved by taking transposes: from what we already proved, we have $A^T \cdot C = \det(A^T) \cdot I_n$, because the cofactor matrix of A^T is C^T. Now, taking transposes, and using $\det(A^T) = \det(A)$, we see that $C^T \cdot A = (A^T \cdot C)^T = (\det(A) \cdot I_n)^T = \det(A) \cdot I_n$.

Similarly to how the first half of this theorem encodes row expansion for determinants, the second half encodes the similar *column expansion*: if you multiply each element of the i-th column by its cofactor and add results, you get the determinant.

A closed formula for the inverse matrix

Theorem. Suppose that $\det(A) \neq 0$. Then

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A).$$

Proof. Indeed, take the formula $A \cdot \operatorname{adj}(A) = \operatorname{adj}(A) \cdot A = \det(A) \cdot I_n$, and divide by $\det(A)$. □

This theorem shows that not only a matrix is invertible when the determinant is not equal to zero, but also that you can compute the inverse by doing exactly one division; all other operations are addition, subtraction, and multiplication.