1111: Linear Algebra I

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Lecture 15

Before the reading week we considered subspaces of $\mathbb{R}^n$. Now our goal is to develop a formalism that would allow to regard these as vector spaces of their own merit.

Abstract vector spaces

Definition 1. An *(abstract) vector space* (over real numbers) is a set V equipped with the following data:

- a rule assigning to each elements $v_1, v_2 \in V$ an element of V denoted $v_1 + v_2$, and
- a rule assigning to each element $v \in V$ and each real number c an element of V denoted $c \cdot v$ (or sometimes cv),

for which the following properties are satisfied:

1. for all $v_1, v_2, v_3 \in V$ we have $(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$,
2. for all $v_1, v_2 \in V$ we have $v_1 + v_2 = v_2 + v_1$,
3. there is a designated *zero element* of V denoted by 0 for which $v + 0 = v$ for all v,
4. for each $v \in V$, there exists $w \in V$, denoted $-v$ and called *the opposite of* v, such that $v + (-v) = 0$,
5. for all $v_1, v_2 \in V$ and all $c \in \mathbb{R}$, we have $c \cdot (v_1 + v_2) = c \cdot v_1 + c \cdot v_2$,
6. for all $c_1, c_2 \in \mathbb{R}$ and all $v \in V$, we have $(c_1 + c_2) \cdot v = c_1 \cdot v + c_2 \cdot v$,
7. for all $c_1, c_2 \in \mathbb{R}$ and all $v \in V$, we have $c_1 \cdot (c_2 \cdot v) = (c_1 c_2) \cdot v$,
8. for all $v \in V$, we have $1 \cdot v = v$.

Note that without the last property, we can have $c \cdot v = 0$ for all v and all c, which leads to some meaningless examples.

Examples of vector spaces

To demonstrate that some set V has a structure of a vector space, we therefore must

- exhibit the rules $v_1, v_2 \mapsto v_1 + v_2$ and $c, v \mapsto c \cdot v$ for $v, v_1, v_2 \in V$, $c \in \mathbb{R}$,
- exhibit the zero element $0 \in V$,
- exhibit, for each $v \in V$, its opposite $-v$,

so that the properties 1-8 hold.

Example 1. Of course, the coordinate vector space $\mathbb{R}^n$, as it says on the tin, is an example of a vector space, with the usual operations on vectors, the usual zero vector, and the opposite vector $-v = (-1) \cdot v$. Properties 1-8 hold, we discussed them on some occasions in the past.

Example 2. Slightly more generally, for given m, n, the set of all $m \times n$-matrices is a vector space with respect to addition and multiplication by scalars. This example is not that different from $\mathbb{R}^n$, we just choose to write numbers not in a column of height mn, but in a rectangular array.

Example 3. Every subspace of $\mathbb{R}^n$ is a vector space. It is *almost* obvious from the definition. Indeed, the definition says that a subspace $U \subset \mathbb{R}^n$ is closed under addition and re-scaling, so properties 1, 2, 5, 6, 7, and 8 hold because they hold for vector operations in $\mathbb{R}^n$. The only things which are not automatic is to check that U contains 0 (for property 3), and that the opposite of every vector of U is in U (for property 4). However, both of these are easy: since U is non-empty, it contains some vector u, and hence it must contain $0 \cdot u = 0$. Also, as we mentioned above, in $\mathbb{R}^n$ we have $-u = (-1) \cdot u$, so the negative of every element of U is in U. Hence properties 3 and 4 hold in U because they hold in $\mathbb{R}^n$.

Example 4. The set $C([0, 1])$ of all continuous functions on the segment $[0, 1]$ is a vector space, with obvious operations $(f+g)(x) = f(x)+g(x)$ and $(c\cdot f)(x) = cf(x)$. The only nontrivial thing (which you either know or will soon know from your analysis module) is that these operations turn continuous functions into continuous functions. All the properties 1-8 are obvious.

Consequences of properties of vector operations

The properties 1-8 altogether allow to operate with elements of V as though they were vectors in $\mathbb{R}^n$, that is create linear combinations, take summands in an equation from the left hand side to the right hand side with opposite signs, collect similar terms etc. For that reason, we shall refer to elements of an abstract vector space as vectors, and to real numbers as scalars.

These properties also allow to prove various theoretical statements about vectors. There will be some of those in your next homework, and for now let me give several examples.

Lemma 1. *If we have $v + a = v$ for some v, then $a = 0$.*

Proof. Suppose that a vector a is such that $v + a = v$ for some v. Then we use $-v$ to move v to the other side of the equation:

$$0 = (-v) + v = (-v) + (v + a) = ((-v) + v) + a = 0 + a = a$$

(by properties 2, 4, 1, 4, 2, and 3), so $a = 0$. □

Lemma 2. *For all $v \in V$, we have $0 \cdot v = 0$.*

Proof. Denote $u = 0 \cdot v$. We have $u + u = 0 \cdot v + 0 \cdot v = (0 + 0 \cdot v) = 0 \cdot v = u$ (we used property 6 in the middle equality). Now we use the previous lemma with $a = v = u$. □

The following lemma proved similarly with property 5 instead of property 6:

Lemma 3. *For all $c \in \mathbb{R}$, we have $c \cdot 0 = 0$.*