1111: Linear Algebra I

Dr. Vladimir Dotsenko (Vlad)

Lecture 17

Coin weighing problem

This example is a question where the viewpoint of linear algebra, as well as using different scalars in vector spaces, turns out to be very important. In fact, I am not aware of more "elementary" solutions.

> Given 101 coins of various shapes and denominations, one knows that if you remove any one coin, the remaining 100 coins can be divided into two groups of 50 of equal total weight. Show that all the coins are of the same weight.

Let us prove this in several steps. We note that if $x_1, \ldots, x_{101}$ are weights of the coins satisfying our assumption, then $x_1 - k$, ..., $x_{101} - k$ are weights that also satisfy our assumption, and lx_1, ..., lx_{101} are weights that also satisfy our assumption, for all k and l.

Let us suppose that weights are not all equal to each other.

First, we consider the case when all weights $x_1, \ldots, x_{101}$ of all coins are integers.

Lemma 1. *The weights of the coins are either all even or all odd.*

Proof. Denote $S = x_1 + \cdots + x_{101}$. Then $S - x_i$ is divisible by 2 for all i, because we can split all coins except for the coin number i into two groups of equal total weight, so $S - x_i$ is twice that weight. Therefore, $x_i - x_j = (S - x_j) - (S - x_i)$ is divisible by 2 also. □

Still assuming that weights are integers, let us note that we can subtract the weight of the first coin of all of them, so get a set of coins with one weight equal to zero. By Lemma, all of the weights are even. We can divide by 2, and still have the coins satisfying our assumption with one weight equal to zero. By Lemma, all of the weights are still even, so we can divide by 2 again, etc. Clearly, the only way to have the weights integer and infinitely divisible by 2 is to have all of them equal to zero, which means that all the weights were equal to each other in the first place.

Now, we suppose all weights are rational. Then, multiplying by common denominator, we get a set of coins satisfying our assumptions where all weights are integers, which we already investigated.

Finally, suppose weights are arbitrary real numbers. Note that the conditions we impose can be expressed as a system of linear equations with rational coefficients! Saying that there is a solution where not all weights are equal is essentially saying that if we adjoin an extra equation $x_1 = 1$, there is a solution where not all coordinates are equal to 1, so this system of equations has at least 2 solutions. But this is a property that "does not depend on scalars", — whether we view our system of equations as a system with rational coefficients or with real coefficients, we do the same, compute the reduced row echelon form. If there are at least two solutions, there must be free unknowns! Setting all these free unknowns equal to zero, we shall obtain a solution with rational coordinates where not all coordinates are equal. But we already proved that the latter was impossible.

Linear independence, span, basis

By definition of a vector space, we can form arbitary linear combinations: if $v_1, \dots, v_k$ are vectors and $c_1, \dots, c_k$ are scalars, then $c_1 v_1 + \dots + c_k v_k$ is a vector which is called the linear combination of $v_1, \dots, v_k$ with coefficients $c_1, \dots, c_k$.

All the definitions that we gave in the case of $\mathbb{R}^n$ proceed in the same way. Below we assume that V is a vector space over real numbers (but one can use any other field if necessary).

Definition 1. A system of vectors $v_1, \dots, v_k \in V$ is said to be linearly independent if the only linear combination of these vectors that is equal to zero is the combination where all the coefficients are equal to zero.

In the next homework, you will have to prove that if $c \cdot v = 0$ then $c = 0$ or $v = 0$. This can be rephrased as follows: one non-zero vector is always linearly independent.

Definition 2. The linear span of vectors $v_1, \dots, v_k \in V$ is the set of all linear combinations of these vectors,

$$\operatorname{span}(v_1, \dots, v_k) = \{c_1 v_1 + \dots + c_k v_k \colon c_1, \dots, c_k \in \mathbb{R}\} .$$

The following statement is easy to check, and is left as an exercise.

Lemma 2. *For any vectors $v_1, \dots, v_k$, the set $\operatorname{span}(v_1, \dots, v_k)$ is a subspace of V.*

Definition 3. A system of vectors $v_1, \dots, v_k \in V$ is said to be *complete*, or to *span* V, if $\operatorname{span}(v_1, \dots, v_k) = V$.

Definition 4. A system of vectors $v_1, \dots, v_k \in V$ is said to form a *basis* of V, if it is linearly independent and spans V.

Example 1. The spanning set that we constructed for the solution set of an arbitrary system of linear equations was, as we remarked, linearly independent, so in fact it provided a basis of that vector space.

Example 2. The *monomials* x^k, $k \geqslant 0$, form a basis in the space of polynomials in one variable. Note that this basis is infinite, but we nevertheless only consider finite linear combinations at all stages. (That is the fundamental difference between linear algebra and functional analysis where convergent infinite series in functional spaces are permitted.)

Dimension

Note that in $\mathbb{R}^n$ we proved that a linearly independent system of vectors consists of at most n vectors, and a complete system of vectors consists of at least n vectors. In a general vector space V, there is no *a priori* n that can play this role. Moreover, the previous example shows that sometimes, no n bounding the size of a linearly independent system of vectors may exist. It however is possible to prove a version of those statements which is valid in every vector space. We shall state it now, and prove next week.

Theorem 1. *Let V be a vector space, and suppose that $e_1, \dots, e_k$ is a linearly independent system of vectors and that $f_1, \dots, f_m$ is a complete system of vectors. Then $k \leqslant m$.*