# 1111: Linear Algebra I

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Lecture 22

In the example we considered in the previous class, one of the two matrices is much simpler than the other ones: the first matrix is diagonal, which simplifies all sorts of computations. We already mentioned that for linear operators we cannot simplify a matrix as much as for linear maps. It is natural to ask whether we may make the corresponding matrix diagonal. Let us show that it is not possible in general. Consider the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. If there exists an invertible matrix $C$ for which $C^{-1}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}C = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$, we can compare traces on the right and on the left, getting $a_1 + a_2 = 2$, $a_1a_2 = 1$, so $a_1$ and $a_2$ are roots of the equation $x^2 - 2x + 1 = 0$, that is $a_1 = a_2 = 1$. But the diagonal matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity matrix, and it represents the map that does not change any vector, so it is the same in any coordinate system, a contradiction.

However, this turns out to be the only source of obstacles. Before we proceed with that, let us introduce a linear algebra context for Fibonacci numbers.

Fibonacci numbers are defined recursively: $f_0 = 0$, $f_1 = 1$, $f_{n+1} = f_n + f_{n-1}$ for $n \geqslant 1$, so that this sequence starts like this:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$$

We shall now explain how to derive a formula for these using linear algebra.

**Idea 1**: let us consider a much simpler question: let $g_0 = 1$, and $g_n = ag_{n-1}$ for $n \geqslant 1$. Then of course $g_n = a^n$.

In our case, each of the numbers is determined by two previous ones, let us store pairs! We put $v_n = \begin{pmatrix} f_n \\ f_{n+1} \end{pmatrix}$.

Then

$$v_n = \begin{pmatrix} f_n \\ f_{n+1} \end{pmatrix} = \begin{pmatrix} f_n \\ f_n + f_{n-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} f_{n-1} \\ f_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}v_{n-1},$$

therefore

$$v_n = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}v_{n-1} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}v_{n-2} = \cdots = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n v_0 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Therefore, we shall be able to compute Fibonacci numbers if we can compute the $n$-th power of the matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$.

**Idea 2**: What does it mean for a matrix of a linear operator $\varphi$ to be diagonal in the system of coordinates given by the basis $e_1, e_2$? This means $\varphi(e_1) = a_1e_1$, $\varphi(e_2) = a_2e_2$. Let us state the relevant general definitions, and then return to our specific question.

**Definition 1.** For a linear operator $\varphi\colon V \to V$, a nonzero vector $v$ satisfying $\varphi(v) = c \cdot v$ for some scalar $c$ is called an *eigenvector* of $\varphi$. The number $c$ is called an *eigenvalue* of $\varphi$.

There is the following general important result.

**Lemma 1.** *Let $\varphi$ be a linear operator. There exists a basis $e_1, \ldots, e_n$ relative to which $\varphi$ has a diagonal matrix if and only if there exists a basis consisting of eigenvectors of $\varphi$.*

The proof is trivial: the two statements literally mean the same. But conceptually, there is an important difference: instead of looking for the change of basis arbitrarily, we should look for eigenvectors. For that, we need a convenient criterion for eigenvalues.

**Lemma 2.** *Let $\varphi$ be a linear operator, and let $A$ be the matrix of $\varphi$ relative to some basis $e_1, \ldots, e_n$. A number $c$ is an eigenvalue of $\varphi$ if and only if $\det(A - cI_n) = 0$.*

*Proof.* Suppose that $c$ is an eigenvalue, which happens if and only if there exists a nonzero vector $v$ such that $\varphi(v) = c \cdot v$. In coordinates relative to the appropriate basis, $A \cdot v_e = c \cdot v_e$, or, in other words, $(A - cI_n) \cdot v_e = 0$. Therefore, $c$ is an eigenvalue if and only if the system of equations $(A - cI_n) \cdot x = 0$ has a nontrivial solution, which happens if and only if the matrix $A - cI_n$ is not invertible, which happens if and only if $\det(A - cI_n) = 0$. $\square$

In general, to determine whether some matrix can be made diagonal by a change of basis, we should write the equation $\det(A - cI_n) = 0$, find all eigenvalues, and then find the corresponding eigenvectors and see if we can find enough of them to form a basis.

In our case, $\det(A - cI_2) = 0$ is the equation $c^2 - c - 1 = 0$ so the eigenvalues are $\frac{1 \pm \sqrt{5}}{2}$.

The corresponding eigenvectors are obtained from solutions of the systems of equations $Ax = \frac{1\pm\sqrt{5}}{2}x$. The first of them has the general solution $\begin{pmatrix} x_1 \\ \frac{1+\sqrt{5}}{2}x_1 \end{pmatrix}$, and the second one has the general solution $\begin{pmatrix} x_1 \\ \frac{1-\sqrt{5}}{2}x_1 \end{pmatrix}$. Setting in each cases $x_1 = 1$, we obtain two eigenvectors $\mathbf{e}_1 = \begin{pmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{pmatrix}$ and $\mathbf{e}_2 = \begin{pmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{pmatrix}$. The transition matrix from the basis of standard unit vectors $\mathbf{s}_1$, $\mathbf{s}_2$ to this basis is, manifestly, $M_{\mathbf{s},\mathbf{e}} = \begin{pmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{pmatrix}$, so

$$M_{\mathbf{s},\mathbf{e}}^{-1} = -\frac{1}{\sqrt{5}} \begin{pmatrix} \frac{1-\sqrt{5}}{2} & -1 \\ \frac{-1-\sqrt{5}}{2} & 1 \end{pmatrix},$$

Since $A\mathbf{e}_1 = (\frac{1+\sqrt{5}}{2})\mathbf{e}_1$, and $A\mathbf{e}_2 = (\frac{1-\sqrt{5}}{2})\mathbf{e}_2$, the matrix of the linear transformation $\varphi$ relative to the basis $\mathbf{e}_1$, $\mathbf{e}_2$ is

$$M_{\mathbf{s},\mathbf{e}}^{-1} A M_{\mathbf{s},\mathbf{e}} = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix}.$$

Therefore,

$$A = M_{\mathbf{s},\mathbf{e}} \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix} M_{\mathbf{s},\mathbf{e}}^{-1},$$

and hence

$$A^n = \left( M_{\mathbf{s},\mathbf{e}} \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix} M_{\mathbf{s},\mathbf{e}}^{-1} \right)^n = M_{\mathbf{s},\mathbf{e}} \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix}^n M_{\mathbf{s},\mathbf{e}}^{-1} = M_{\mathbf{s},\mathbf{e}} \begin{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n \end{pmatrix} M_{\mathbf{s},\mathbf{e}}^{-1}.$$

Substituting the above formulas for $M_{\mathbf{s},\mathbf{e}}$ and $M_{\mathbf{s},\mathbf{e}}^{-1}$, we see that

$$A^n = \begin{pmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n \end{pmatrix} - \frac{1}{\sqrt{5}} \begin{pmatrix} \frac{1-\sqrt{5}}{2} & -1 \\ \frac{-1-\sqrt{5}}{2} & 1 \end{pmatrix}$$

In fact, we have $\mathbf{v}_n = A^n \mathbf{v}_0$, so

$$\mathbf{v}_n = \begin{pmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n \end{pmatrix} \left(-\frac{1}{\sqrt{5}}\right) \begin{pmatrix} \frac{1-\sqrt{5}}{2} & -1 \\ \frac{-1-\sqrt{5}}{2} & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n \\ -\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right) \\ \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right) \end{pmatrix}$$

Recalling that $\mathbf{v}_n = \begin{pmatrix} f_n \\ f_{n+1} \end{pmatrix}$, we observe that

$$f_n = -\left(\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1-\sqrt{5}}{2}\right)^n\right).$$

As a little remark, since $\left|\frac{1-\sqrt{5}}{2}\right| < 1$, for large $n$ the $n$-th Fibonacci number is just the closest integer to $\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$.