MA 1112: Linear Algebra II
Selected answers/solutions to the assignment for March 11, 2019

1. The characteristic polynomial of

$$\begin{pmatrix} -3 & -1 & 0 \\ -1 & -1 & 0 \\ -1 & -2 & 1 \end{pmatrix},$$

is $-t^3 - 3t^2 + 2t + 2 = -(t-1)(t^2 + 4t + 2)$, the characteristic polynomial of

$$\begin{pmatrix} 9 & 5 & 2 \\ -16 & -9 & -4 \\ 2 & 1 & 1 \end{pmatrix},$$

is $-t^3 + t^2 + t - 1 = -(t-1)^2(t+1)$, the characteristic polynomial of

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

is $(t-1)^3$, the characteristic polynomial of

$$\begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix},$$

is $(t-2)^3$, the characteristic polynomial of

$$\begin{pmatrix} -2 & -4 & 16 \\ 0 & 2 & 0 \\ -1 & -1 & 6 \end{pmatrix},$$

is also $(t-2)^3$, but if we subtract $2I$ from the latter two matrices, we get a matrix of rank 2 and a matrix of rank 1, so they are not similar. Finally,

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

has the characteristic polynomial $-(t-1)^2(t+1)$, which is the same as the one of the second matrix. The eigenvalue -1 leads to just one block of size 1. To investigate the what happens for the eigenvalue 1, we subtract the identity matrix from each of those matrices. The result is of rank 2, so each of these matrices has a Jordan block of size two for that eigenvalue. Thus, the two matrices have the same Jordan form and are similar.

2. (**a**) The characteristic polynomial of B is $(t-2)^2$, so both eigenvalues are equal to 2. We have $B - 2I = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$, and it is immediate to see that every eigenvector is proportional to $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

(**b**) We should have one thread of length 2, so we start with a vector not proportional to $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, for example, $v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ compensating for the missing pivot. Then $(B-2I)v = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, and these two matrices form a Jordan basis. The matrix C is $\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$.

(**c**) We have $C^{-1}BC = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$, so $B = C\begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}C^{-1}$ and

$$B^n = C\begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}^n C^{-1} = C\begin{pmatrix} 2^n & 0 \\ n2^{n-1} & 2^n \end{pmatrix}^n C^{-1} = \begin{pmatrix} 2^n - n2^{n-1} & -n2^{n-1} \\ n2^{n-1} & 2^n + n2^{n-1} \end{pmatrix}.$$

Finally, if we let $v_k = \begin{pmatrix} x_k \\ y_k \end{pmatrix}$, then $v_{k+1} = Bv_k$, so $v_n = B^n v_0 = \begin{pmatrix} 2^n + 4n2^{n-1} \\ -52^n - 4n2^{n-1} \end{pmatrix}$.

3. Let $v_k = \begin{pmatrix} a_k \\ a_{k+1} \end{pmatrix}$, then $v_{k+1} = Bv_k$, where $B = \begin{pmatrix} 0 & 1 \\ -49 & 14 \end{pmatrix}$. Both eigenvalues of B are equal to 7. We take the vector $v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ outside the kernel of $B - 7I$ that would compensate for the missing pivot; we have $(B - 7I)v = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$, so the columns of $C = \begin{pmatrix} 0 & 1 \\ 1 & 7 \end{pmatrix}$ form a Jordan basis. Thus, $C^{-1}BC = \begin{pmatrix} 7 & 0 \\ 1 & 7 \end{pmatrix}$, and

$$B^n = C\begin{pmatrix} 7 & 0 \\ 1 & 7 \end{pmatrix}^n C^{-1} = \begin{pmatrix} 7^n - n7^n & n7^{n-1} \\ -n7^{n+1} & 7^n + n7^n \end{pmatrix}.$$

Finally, $v_n = B^n \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -7^n + n7^n + n7^{n-1} \\ n7^{n+1} + 7^n + n7^n \end{pmatrix}$, so $a_n = -7^n + n7^n + n7^{n-1}$.

4. (**a**) Expanding A_n along the first row, and then expanding the second cofactor along the first column, we get precisely the recurrent relation $\det(A_n) = 5\det(A_{n-1}) - 4\det(A_{n-2})$. Note that if we put $\det(A_0) = 1$, this formula is valid for $n = 2$ as well.

(**b**) For brevity, denote $a_n = \det(A_n)$, $v_n = \begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix}$. Then $v_{n+1} = Av_n$, where $A = \begin{pmatrix} 0 & 1 \\ -4 & 5 \end{pmatrix}$, so $v_n = A^n v_0 = A^n \begin{pmatrix} 1 \\ 5 \end{pmatrix}$. The eigenvalues of A are 1 and 4, with the corresponding eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$. Thus, if $C = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}$, we have $C^{-1}AC = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$, and

$$A^n = C\begin{pmatrix} 1 & 0 \\ 0 & 4^n \end{pmatrix} C^{-1} = \begin{pmatrix} \frac{4-4^n}{3} & \frac{4^n-1}{3} \\ \frac{4-4^{n+1}}{3} & \frac{4^{n+1}-1}{3} \end{pmatrix}$$

. Finally, $v_n = A^n v_0 = \begin{pmatrix} \frac{4^{n+1}-1}{3} \\ \frac{4^{n+2}-1}{3} \end{pmatrix}$, and $a_n = \frac{4^{n+1}-1}{3}$.