MA 1112: Linear Algebra II
Selected answers/solutions to the assignment for March 25, 2019

1. We have
$(\mathbf{v},\mathbf{v}) = \int_{-1}^{1}(t^2-t-1)^2\,dt = \int_{-1}^{1}(t^4-2t^3-t^2+2t+1)\,dt = 26/15$,
$(\mathbf{v},\mathbf{w}) = \int_{-1}^{1}(t^2-t-1)(t^3+t^2+t+1)\,dt = \int_{-1}^{1}(t^5-t^3-t^2-2t-1)\,dt = -8/3$,
$(\mathbf{w},\mathbf{w}) = \int_{-1}^{1}(t^3+t^2+t+1)^2\,dt = \int_{-1}^{1}(t^6+2t^5+3t^4+4t^3+3t^2+2t+1)\,dt = 192/35$, so the length of $\mathbf{v}$ is $\sqrt{26/15}$ and the angle between $\mathbf{v}$ and $\mathbf{w}$ is

$$\arccos\frac{-8/3}{\sqrt{26/15\cdot 192/35}} = \arccos(-5/\sqrt{78}).$$

2. The orthogonal complement of our subspace consists of all vectors which are orthogonal to both of the spanning vectors, that is vectors $\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix}$ for which $x_1+x_2+x_3+x_4=0$ and $x_1-x_3+x_5=0$. Solving this system (the variables x_3,x_4,x_5 are free), we get a parametrisation of the orthogonal complement: $\begin{pmatrix}u-w\\w-2u-v\\u\\v\\w\end{pmatrix}$, where $u,v,w\in\mathbb{R}$.

3. This function product is clearly bilinear. Also, it is symmetric because

$$\operatorname{tr}(BA^T) = \operatorname{tr}((BA^T)^T) = \operatorname{tr}(AB^T).$$

Finally, it is positive definite because the trace of AA^T is equal to the sum of squares of all matrix elements of A. If A is symmetric and B is skew-symmetric, then $(A,B) = \operatorname{tr}(AB^T) = -\operatorname{tr}(AB)$ and $(B,A) = \operatorname{tr}(BA^T) = \operatorname{tr}(BA)$, but since $\operatorname{tr}(AB) = \operatorname{tr}(BA)$, we conclude that $\operatorname{tr}(AB) = -\operatorname{tr}(AB)$, so $\operatorname{tr}(AB) = 0$, and $(A,B) = 0$.

4. The matrix of the corresponding bilinear form is

$$A = \begin{pmatrix}2 & a & 1\\ a & 1 & 1-a\\ 1 & 1-a & 1\end{pmatrix}.$$

We have $\Delta_1 = 2$, $\Delta_2 = 2-a^2$, $\Delta_3 = -5a^2+6a-1$. All these numbers are positive if and only if $|a| < \sqrt{2}$ and $1/5 < a < 1$ (since the roots of $-5a^2+6a-1$ are $1/5$ and 1). In fact, the second condition implies the first one, so we get the answer $1/5 < a < 1$.

5. (**a**) $\Delta_1 = 2$, $\Delta_2 = 3$, $\Delta_3 = 4$, so by Sylvester's criterion the signature is $(3,0,0)$.

(**b**) $\Delta_1 = 1$, $\Delta_2 = -2$, $\Delta_3 = 3$, so by Jacobi's theorem the signature can be read from the sequence $1/1$, $1/(-2)$, $-2/3$; it is $(1,2,0)$.

(**c**) $\Delta_1 = -1$, $\Delta_2 = 1$, $\Delta_3 = 7$, so by Jacobi's theorem the signature can be read from the sequence $1/(-1)$, $-1/1$, $1/7$; it is $(1,2,0)$.

(**d**) $\Delta_1 = -1$, $\Delta_2 = 1$, $\Delta_3 = 2$, $\Delta_4 = -34$, so by Jacobi's theorem the signature can be read from the sequence $1/(-1)$, $-1/1$, $1/2$, $-2/34$; it is $(1,3,0)$.