MA 1112: Linear Algebra II
Selected answers/solutions to the assignment for April 1, 2019

1. First of all, for the last term $\sin^3(x_1+x_2+x_3)$ all second derivatives at the origin are clearly equal to zero (this expression is the cube of something that vanishes at the origin; differentiating reduces the order of zero by one, so the second derivatives of the cube are still equal to zero). Thus, from now on we replace f by $g=\sin(x_1-x_2)\sin(x_1-x_3)+\sin^2 x_2+c\sin x_3\sin(2x_3)$. We have
$\frac{\partial g}{\partial x_1}=\sin(x_1-x_2)\cos(x_1-x_3)+\cos(x_1-x_2)\sin(x_1-x_3)=\sin(x_1-x_2+x_1-x_3)=\sin(2x_1-x_2-x_3)$,
$\frac{\partial g}{\partial x_2}=-\cos(x_1-x_2)\sin(x_1-x_3)+2\sin(x_2)\cos(x_2)=-\cos(x_1-x_2)\sin(x_1-x_3)+\sin(2x_2)$,

$$\frac{\partial g}{\partial x_3}=-\sin(x_1-x_2)\cos(x_1-x_3)+c(\cos(x_3)\sin(2x_3)+2\sin(x_3)\cos(2x_3)),$$

$\frac{\partial^2 g}{\partial x_1^2}=2\cos(2x_1-x_2-x_3)$,
$\frac{\partial^2 g}{\partial x_1\partial x_2}=-\cos(2x_1-x_2-x_3)$,
$\frac{\partial^2 g}{\partial x_1\partial x_3}=-\cos(2x_1-x_2-x_3)$,
$\frac{\partial^2 g}{\partial x_2\partial x_1}=-\cos(2x_1-x_2-x_3)$,
$\frac{\partial^2 g}{\partial x_2^2}=-\sin(x_1-x_2)\sin(x_1-x_3)+2\cos(2x_2)$,
$\frac{\partial^2 g}{\partial x_2\partial x_3}=\cos(x_1-x_2)\cos(x_1-x_3)$,
$\frac{\partial^2 g}{\partial x_3\partial x_1}=-\cos(2x_1-x_2-x_3)$,
$\frac{\partial^2 g}{\partial x_3\partial x_2}=\cos(x_1-x_2)\cos(x_1-x_3)$,

$$\frac{\partial^2 g}{\partial x_3^2}=-\sin(x_1-x_2)\sin(x_1-x_3)+c(2\cos(2x_3)\cos(x_3)-$$
$$-\sin(x_3)\sin(2x_3)+2\cos(2x_3)\cos(x_3)-4\sin(x_3)\sin(2x_3)),$$

so
$$A=\begin{pmatrix}2&-1&-1\\-1&2&1\\-1&1&4c\end{pmatrix}.$$

Top left corner determinants of this matrix are 2, 3, and $12c-2$, so by Sylvester's criterion the corresponding quadratic form is positive definite for $c>\frac{1}{6}$.

Also, since all first derivatives of f and of g vanish at the origin, we have

$$f(x+h)=f(x)+A(h,h)+\varepsilon(h),$$

where $\lim\limits_{h\to 0}\frac{\varepsilon(h)}{|h|^2}=0$. For $c=1/5>1/6$ our form is positive definite, so f has a local minimum at the origin.

2. (**a**) There are many different ways to solve this problem. For example, if we denote by $x_1,\ldots,x_n$ the respective coordinates, then the associated quadratic form is

$$x_1^2+x_1x_2+\ldots+x_1x_n+x_2x_1+x_2^2+\ldots+x_2x_n+\ldots+x_n^2=(x_1+x_2+\ldots+x_n)^2,$$

so we need just one square, and the signature is $(1,0,n-1)$. (The number of zeros is $n-1$ because the total number of coordinates is n.)

Another approach, which we shall need in the next part of the question anyway, is to find eigenvalues of this matrix. This matrix is obviously of rank 1, so it has a kernel of dimension $n-1$. The kernel is the subspace of eigenvectors with eigenvalue 0, so we already know that $n-1$ of the n eigenvalues of our matrix are equal to 0. To find the last eigenvalue, we note that the sum of eigenvalues is equal to the trace, that is equal to n in our case, so the only nonzero eigenvalue is n. Overall, the signature is $(1,0,n-1)$, since we have one positive eigenvalue, no negative ones, and $n-1$ zero eigenvalues.

Just for the record, the following vectors give an orthogonal basis where the matrix becomes diagonal (even though for the signature we do not need it):

$$\begin{pmatrix}1\\-1\\0\\0\\\vdots\\0\\0\end{pmatrix},\begin{pmatrix}1\\1\\-2\\0\\\vdots\\0\\0\end{pmatrix},\begin{pmatrix}1\\1\\1\\-3\\\vdots\\0\\0\end{pmatrix},\dots,\begin{pmatrix}1\\1\\1\\1\\\vdots\\1\\-n+1\end{pmatrix},\begin{pmatrix}1\\1\\1\\1\\\vdots\\1\\1\end{pmatrix}.$$

(**b**) The matrix of this bilinear form is obtained from the matrix of the previous question by subtracting the identity matrix. Hence, its eigenvalues are obtained from the eigenvalues from the previous question by subtracting 1 from each of them, that is they are $n-1$, -1, -1, $\dots$, -1, and the signature is $(1,n-1,0)$.

3. Eigenvalues are 0, 1, and 3, so the corresponding eigenvectors are orthogonal. Normalizing them (dividing by lengths), we get an orthonormal basis of eigenvectors

$$\frac{1}{\sqrt{3}}\begin{pmatrix}-1\\1\\1\end{pmatrix},\frac{1}{\sqrt{2}}\begin{pmatrix}0\\-1\\1\end{pmatrix},\frac{1}{\sqrt{6}}\begin{pmatrix}2\\1\\1\end{pmatrix}.$$

We have

$$\max_{(x,x)=1} q(x)=3,\quad \min_{(x,x)=1} q(x)=0,$$

as these values are given by the max/min eigenvalue of our matrix. The minimum is reached on the first vector from the basis of eigenvectors, the maximum — on the third vector.

4. We are looking for a basis of the form

$$\begin{aligned}f_1&=\alpha_{11}e_1,\\ f_2&=\alpha_{12}e_1+\alpha_{22}e_2,\\ f_3&=\alpha_{13}e_1+\alpha_{23}e_2+\alpha_{33}e_3,\end{aligned}$$

imposing equations $A(e_i,f_j)=0$ for $i<j$, and $A(e_i,f_i)=1$ for all i. This means that

$$\begin{aligned}1&=A(e_1,f_1)=2\alpha_{11},\\ 0&=A(e_1,f_2)=2\alpha_{12}+\alpha_{22},\\ 1&=A(e_2,f_2)=\alpha_{12}+2\alpha_{22},\\ 0&=A(e_1,f_3)=2\alpha_{13}+\alpha_{23},\\ 0&=A(e_2,f_3)=\alpha_{13}+2\alpha_{23}+\alpha_{33},\\ 1&=A(e_3,f_3)=\alpha_{23}+2\alpha_{33}.\end{aligned}$$

Solving these linear equations, we get $\alpha_{11}=\frac{1}{2}$, $\alpha_{12}=-\frac{1}{3}$, $\alpha_{22}=\frac{2}{3}$, $\alpha_{13}=\frac{1}{4}$, $\alpha_{23}=-\frac{1}{2}$, $\alpha_{33}=\frac{3}{4}$, so the required change of basis is

$$\begin{aligned}f_1&=\frac{1}{2}e_1,\\ f_2&=-\frac{1}{3}e_1+\frac{2}{3}e_2,\\ f_3&=\frac{1}{4}e_1-\frac{1}{2}e_2+\frac{3}{4}e_3.\end{aligned}$$