MA 1112: Linear Algebra II
Homework problems for January 28, 2019

Solutions to this problem sheet are to be handed in after our class at 11am on Monday. Please attach a cover sheet with a declaration `http://tcd-ie.libguides.com/plagiarism/declaration` confirming that you know and understand College rules on plagiarism. On the same cover sheet, please put your name, student number, and name of the degree (Maths/TP/TSM), and staple all the sheets together. (Failure to do that may result in misplaced/lost sheets, for which no responsibility can be taken by instructors.)

1. For each of the following matrices A, viewed as a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$,

- compute the rank of A;
- describe all eigenvalues and eigenvectors of A;
- determine whether there exists a change of coordinates making the matrix A diagonal.

(**a**) $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$; (**b**) $A = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 4 \\ 1 & -1 & 4 \end{pmatrix}$; (**c**) $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & 5 & 2 \end{pmatrix}$.

2. Compute the rank of the matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

3. Let V be the space of all polynomials in t of degree at most n, and let $\varphi\colon V \to V$ be the linear transformation given by $(\varphi(p(t)) = p'(t) + 3p''(t)$. Find the eigenvalues and the eigenvectors of φ, and show that there is no basis of V relative to which the matrix of φ is diagonal. (*Hint*: first show that φ has no non-zero eigenvalues.)

4. Prove that for every $n \times m$-matrix A of rank 1 there exist an $n \times 1$-matrix $B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$ and a $1 \times n$-matrix $C = (c_1, c_2, \ldots, c_m)$ such that

$$A = BC = \begin{pmatrix} b_1c_1 & b_1c_2 & \ldots & b_1c_m \\ b_2c_1 & b_2c_2 & \ldots & b_2c_m \\ \vdots & \ddots & \ldots & \vdots \\ b_nc_1 & b_nc_2 & \ldots & b_nc_m \end{pmatrix},$$

(*Hint*: denote by B a nonzero column of the matrix A, and find an appropriate C.)

5. Let $\alpha\colon U \to V$ and $\beta\colon V \to W$ be linear maps of vector spaces. Prove that $\mathrm{rk}(\beta \circ \alpha) \leqslant \mathrm{rk}(\alpha)$ and $\mathrm{rk}(\beta \circ \alpha) \leqslant \mathrm{rk}(\beta)$. (As always, $\beta \circ \alpha$ denotes the composite map $(\beta \circ \alpha)(u) = \beta(\alpha(u))$.)