MA1112: Linear Algebra II

Dr. Vladimir Dotsenko (Vlad)

Lecture 10

Jordan decomposition theorem

Combining the results we proved, we establish the following key result.

Jordan decomposition theorem. Let V be a finite-dimesional vector space over $\mathbb{C}$. For a linear transformation $\varphi\colon V \to V$, there exists a basis of V of the form

$$\begin{gathered} e_1^{(1)}, \ldots, e_{m_1}^{(1)}, \\ e_1^{(2)}, \ldots, e_{m_2}^{(2)}, \\ \ldots \\ e_1^{(s)}, \ldots, e_{m_s}^{(s)} \end{gathered}$$

and scalars λ_1, ..., λ_s such that

$$\begin{gathered} (\varphi - \lambda_i I)e_1^{(i)} = e_2^{(i)}; \\ (\varphi - \lambda_i I)e_2^{(i)} = e_3^{(i)}; \\ \ldots \\ (\varphi - \lambda_i I)e_{m_i}^{(i)} = 0 \end{gathered}$$

With respect to this basis, the matrix of φ has a block-diagonal matrix made of blocks

$$J_{m_i}(\lambda) = \begin{pmatrix} \lambda_i & 0 & 0 & 0 & \ldots & 0 & 0 \\ 1 & \lambda_i & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \lambda_i & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & \ldots & \lambda_i & 0 \\ 0 & 0 & 0 & 0 & \ldots & 1 & \lambda_i \end{pmatrix},$$

a block $J_{m_i}(\lambda_i)$ for a thread of length m_i. Indeed, on each individual subspace U_i, we consider the linear transformation $\varphi_\lambda = \varphi - \lambda_i I$ which is nilpotent on that subspace. Therefore, our previous results allow us to find a basis of threads for this linear transformation, and its matrix is block-diagonal made of blocks

$$J_l = \begin{pmatrix} 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & 0 & \ldots & 1 & 0 \end{pmatrix},$$

one block J_l for each thread of length l. Recalling that $\varphi = B_{\lambda_i} + \lambda_i I$, we obtain the blocks mentioned above.

Examples

From our proof, one sees that for computing the Jordan normal form and a Jordan basis of a linear transformation φ on a vector space V, one can use the following plan:

- Find all eigenvalues of φ (that is, compute the characteristic polynomial $\det(A - cI)$ of the corresponding matrix A, and determine its roots $\lambda_1, \ldots, \lambda_k$).
- For each eigenvalue λ, form the linear transformation $\varphi_\lambda = \varphi - \lambda I$ and consider the increasing sequence of subspaces
$$\text{Ker}\,\varphi_\lambda \subset \text{Ker}\,\varphi_\lambda^2 \subset \ldots$$
and determine where it stabilizes, that is find the smallest number k for which $\text{Ker}\,\varphi_\lambda^k = \text{Ker}\,\varphi_\lambda^{k+1}$. Let $U = \text{Ker}\,\varphi_\lambda^k$. The subspace U is an invariant subspace of φ_λ (and φ), and φ_λ is nilpotent on U, so it is possible to find a basis consisting of several "threads" of the form $f, \varphi_\lambda f, \varphi_\lambda^2 f, \ldots$, where φ_λ shifts vectors along each thread (as in the previous homework).
- Joining all the threads (for different λ) together, we get a Jordan basis for A. A thread of length p for an eigenvalue λ contributes a Jordan block $J_p(\lambda)$ to the Jordan normal form.

Example 1. Let $V = \mathbb{R}^3$, and $A = \begin{pmatrix} -2 & 2 & 1 \\ -7 & 4 & 2 \\ 5 & 0 & 0 \end{pmatrix}$.

The characteristic polynomial of A is $-t + 2t^2 - t^3 = -t(1-t)^2$, so the eigenvalues of A are 0 and 1.

Furthermore, $A - I = \begin{pmatrix} -3 & 2 & 1 \\ -7 & 3 & 2 \\ 5 & 0 & -1 \end{pmatrix}$, $(A-I)^2 = \begin{pmatrix} 0 & 0 & 0 \\ 10 & -5 & -3 \\ -20 & 10 & 6 \end{pmatrix}$, so $\text{rk}(A - I) = 2$, $\text{rk}(A-I)^2 = 1$. Note that $\text{rk}(A) = 2$. This shows that there is at least one thread of length at least 2 for the eigenvalue 1, and at least one thread of length at least 1 for the eigenvalue 0. Since our vector space is three-dimensional, there is nothing else, and kernels of powers stabilize from $(A-I)^2$ for the eigenvalue 1 and from A for the eigenvalue 0.

To determine the basis of $\text{Ker}(A)$, we solve the system $Av = 0$ and obtain a vector $f = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$.

To deal with the eigenvalue 1, we see that the kernel of $A - I$ is spanned by the vector $\begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix}$, the kernel of $(A-I)^2 = \begin{pmatrix} 0 & 0 & 0 \\ 10 & -5 & -3 \\ -20 & 10 & 6 \end{pmatrix}$ is spanned by the vectors $\begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 3/10 \\ 0 \\ 1 \end{pmatrix}$. Reducing the latter vectors using the former one, we end up with the relative basis vector $e = \begin{pmatrix} 0 \\ 3 \\ -5 \end{pmatrix}$, which gives rise to a thread $e, (A-I)e = \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix}$. Overall, a Jordan basis is given by $f, e, (A-I)e$, and the Jordan normal form has a block of size 1 with 0 on the diagonal, and a block of size 2 with 1 on the diagonal:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Example 2. Let $V = \mathbb{R}^4$, and $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 11 & 6 & -4 & -4 \\ 22 & 15 & -8 & -9 \\ -3 & -2 & 1 & 2 \end{pmatrix}$.

The characteristic polynomial of A is $1 - 2t^2 + t^4 = (1+t)^2(1-t)^2$, so the eigenvalues of A are -1 and 1. To avoid unnecessary calculations (similar to avoiding computing $(A - I)^3$ in the previous example), let us compute the ranks for both eigenvalues simultaneously. For $\lambda = -1$ we have $A + I = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 11 & 7 & -4 & -4 \\ 22 & 15 & -7 & -9 \\ -3 & -2 & 1 & 3 \end{pmatrix}$, $\mathrm{rk}(A + I) = 3$, $(A + I)^2 = \begin{pmatrix} 12 & 8 & -4 & -4 \\ 12 & 8 & -4 & -4 \\ 60 & 40 & -20 & -24 \\ -12 & -8 & 4 & 8 \end{pmatrix}$, $\mathrm{rk}((A + I)^2) = 2$.
For $\lambda = 1$ we have $A - I = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 11 & 5 & -4 & -4 \\ 22 & 15 & -9 & -9 \\ -3 & -2 & 1 & 1 \end{pmatrix}$, $\mathrm{rk}(A - I) = 3$, $(A - I)^2 = \begin{pmatrix} 12 & 4 & -4 & -4 \\ -32 & -16 & 12 & 12 \\ -28 & -20 & 12 & 12 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $\mathrm{rk}((A - I)^2) = 2$. This shows that there is at least one thread of length at least 2 for the eigenvalue 1, and at least one thread of length at least 2 for the eigenvalue -1. Since our vector space is four-dimensional, there is nothing else, and kernels of powers stabilize starting from the square for each eigenvalue.

Thus, each of these eigenvalues gives rise to a thread of length at least 2, and since our vector space is 4-dimensional, each of the threads should be of length 2, and in each case the stabilisation happens on the second step.

In the case of the eigenvalue -1, we first determine the kernel of $A + I$, solving the system $(A + I)v = 0$; this gives us a vector $\begin{pmatrix} -1 \\ 1 \\ -1 \\ 0 \end{pmatrix}$. The equations that determine the kernel of $(A + I)^2$ are $t = 0, 3x + 2y = z$ so y and z are free variables, and for the basis vectors of that kernel we can take $\begin{pmatrix} 1/3 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -2/3 \\ 1 \\ 0 \\ 0 \end{pmatrix}$. Reducing the basis vectors of $\mathrm{Ker}(A + I)^2$ using the basis vector of $\mathrm{Ker}(A + I)$, we end up with a relative basis vector $e = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix}$, and a thread $e, (A + I)e = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$.

In the case of the eigenvalue 1, we first determine the kernel of $A - I$, solving the system $(A - I)v = 0$; this gives us a vector $\begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$. The equations that determine the kernel of $(A - I)^2$ are $4x = z + t, 4y = z + t$ so z and t are free variables, and for the basis vectors of that kernel we can take $\begin{pmatrix} 1/4 \\ 1/4 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1/4 \\ 1/4 \\ 0 \\ 1 \end{pmatrix}$. Reducing the basis vectors of $\mathrm{Ker}(A - I)^2$ using the basis vector of $\mathrm{Ker}(A + I)$, we end up with a relative basis vector $f = \begin{pmatrix} 1/4 \\ 1/4 \\ 0 \\ 1 \end{pmatrix}$, and a thread $e, (A - I)e = \begin{pmatrix} 0 \\ 0 \\ 1/4 \\ -1/4 \end{pmatrix}$.

Finally, the vectors $e, (A + I)e, f, (A - I)f$ form a Jordan basis for A; the Jordan normal form of A is $\begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$.