# MA1112: Linear Algebra II

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Lecture 17

Today we shall prove various theorems on quadratic forms stated without proof last week.

**Theorem 1.** *Let $q$ be a quadratic form on a vector space $V$. There exists a basis $f_1,\ldots,f_n$ of $V$ for which the quadratic form $q$ becomes a signed sum of squares:*

$$q(x_1f_1+\cdots+x_nf_n)=\sum_{i=1}^n \varepsilon_i x_i^2,$$

*where all numbers $\varepsilon_i$ are either $1$ or $-1$ or $0$.*

*Proof.* Informally, the slogan behind the proof we shall present is "imitate the Gram–Schmidt procedure". Let us make it precise. We shall argue by induction on $\dim V$, the basis of induction being $\dim V=0$, when the basis is empty, so there is nothing to prove.

Suppose $\dim V=n>0$. There are two cases to consider. First, it might be the case that $q(v)=0$ for all $v$. In this case, any basis would work, with $\varepsilon_i=0$ for all $i$.

Otherwise, there exists a vector $v$ such that $q(v)\neq 0$. Let us extend it to a basis $e_1=v$, $e_2$, ..., $e_n$, and look at the symmetric bilinear form $b(v,w)=\frac12(q(v+w)-q(v)-q(w))$ associated to the quadratic form $q$. We claim that we can change the basis $e_1$, ..., $e_n$ into a basis $e_1',e_2',\ldots,e_n'$ with $e_1'=e_1=v$ and $b(e_i',e_1')=0$ for all $i=2,\ldots,n$. Indeed, we can put $e_i'=e_i-\frac{b(e_i,e_1)}{b(e_1,e_1)}e_1$ for $i>1$; here division by $b(e_1,e_1)$ is possible since $b(e_1,e_1)=q(e_1)=q(v)\neq 0$. We can now consider the linear span of $e_2',\ldots,e_n'$ with the bilinear form $q$, and proceed by induction on dimension. Therefore, we can find a basis $f_2,\ldots,$ $f_n$ of that space with the required property. It remains to note that if we take $e_1=\frac{1}{\sqrt{|q(v)|}}v$, then we have $q(e_1)=\pm1$, and also $b(e_1,e_i)=0$ for $i>1$, which proves that our quadratic form becomes a signed sum of squares in this basis. □

**Theorem 2** (Law of inertia)**.** *In the previous theorem, the triple $(n_+,n_-,n_0)$, where $n_\pm$ is the number of $\varepsilon_i$ equal to $\pm1$, and $n_0$ is the number of $\varepsilon_i$ equal to $0$, does not depend on the choice of the basis $f_1,\ldots,f_n$. This triple is often referred to as the* signature *of the quadratic form $q$.*

*Proof.* Suppose that we have a basis

$$e_1,\ldots,e_{n_+},f_1,\ldots,f_{n_-},g_1,\ldots,g_{n_0}$$

which produces a system of coordinates where $q$ becomes a signed sum of squares with $n_+$ of $\varepsilon_i$ are equal to 1, $n_-$ of $\varepsilon_i$ are equal to $-1$, and $n_0$ of $\varepsilon_i$ are equal to 0. Let us look at the corresponding symmetric bilinear form $b$. The reconstruction formula $b(v,w)=\frac12(q(v+w)-q(v)-q(w))$ implies that

$$b(x_1e_1+\cdots+z_{n_0}g_{n_0},x_1'e_1+\cdots+z_{n_0}'g_{n_0})=x_1x_1'+\cdots+x_{n_+}x_{n_+}'-y_1y_1'-\cdots-y_{n_-}y_{n_-}'.$$

By definition, the *kernel* of a symmetric bilinear form is the space of all vectors $v$ such that $b(v,w)=0$ for all $w\in V$. We see that the kernel is defined by a system of equations $b(v,e_i)=b(v,f_j)=b(v,g_k)=0$ for all $i,j,k$, and by direct inspection this system implies that $v$ is a linear combination of the vectors $g_k$. This implies that $n_0$ is the dimension of the kernel of $b$, and so is independent of any choices.

Suppose now that there are two different bases

$$e_1,\ldots,e_{n_+},f_1,\ldots,f_{n_-},g_1,\ldots,g_{n_0}$$

and

$$e'_1, \dots, e_{n'_+}, f'_1, \dots, f_{n'_-}, g'_1, \dots, g'_{n_0}$$

where $q$ is a signed sum of squares, and $n_+ \neq n'_+$, so without loss of generality $n_+ > n'_+$. Note that this implies that $n_- < n'_-$. Consider the vectors

$$e_1, \dots, e_{n_+}, f'_1, \dots, f'_{n'_-}, g_1, \dots, g_{n_0}.$$

The total number of those vectors exceeds the dimension of $V$, so they must be linearly dependent, that is

$$a_1 e_1 + \dots + a_{n_+} e_{n_+} + b_1 f'_1 + \dots + b_{n'_-} f'_{n'_-} + c_1 g_1 + \dots + c_{n_0} g_{n_0} = 0$$

for some scalars $a_i, b_j, c_k$. Let us rewrite it as

$$a_1 e_1 + \dots + a_{n_+} e_{n_+} + c_1 g_1 + \dots + c_{n_0} g_{n_0} = -(b_1 f'_1 + \dots + b_{n'_-} f'_{n'_-}),$$

and denote the vector to which both the left hand side and the right hand side are equal to by $v$. Then

$$a_1^2 + \dots + a_{n_+}^2 = q(v) = -b_1^2 - \dots - b_{n_-}^2,$$

which implies

$$a_1 = \dots = a_{n_+} = b_1 = \dots = b_{n_-} = 0,$$

and substituting it into

$$a_1 e_1 + \dots + a_{n_+} e_{n_+} + b_1 f'_1 + \dots + b_{n'_-} f'_{n'_-} + c_1 g_1 + \dots + c_{n_0} g_{n_0} = 0$$

we get

$$c_1 g_1 + \dots + c_{n_0} g_{n_0} = 0,$$

implying of course $c_1 = \dots = c_{n_0} = 0$, which altogether shows that these vectors cannot be linearly dependent, a contradiction. □