MA1112: Linear Algebra II

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Lecture 8

Some examples for the case $\varphi^k = 0$

Yesterday, we proved that for every linear transformation $\varphi\colon V \to V$ with $\varphi^k = 0$ for some k, it is possible to choose a basis

$$\begin{gathered} e_1^{(1)}, e_2^{(1)}, e_3^{(1)}, \ldots, e_{n_1}^{(1)}, \\ e_1^{(2)}, e_2^{(2)}, \ldots, e_{n_2}^{(2)}, \\ \ldots \\ e_1^{(l)}, \ldots, e_{n_l}^{(l)} \end{gathered}$$

of V such that for each "thread"

$$e_1^{(p)}, e_2^{(p)}, \ldots, e_{n_p}^{(p)}$$

we have

$$\varphi(e_1^{(p)}) = e_2^{(p)}, \varphi(e_2^{(p)}) = e_3^{(p)}, \ldots, \varphi(e_{n_p}^{(p)}) = 0.$$

Today we shall discuss several examples of computing such bases of threads for a linear transformation.

Example 1. Let us consider the case of $V = \mathbb{R}^3$, where φ is multiplication by the matrix $A = \begin{pmatrix} -3 & 1 & -1 \\ -12 & 4 & -4 \\ -3 & 1 & -1 \end{pmatrix}$.

We have $A^2 = 0$, so $\varphi^2 = 0$, falling into the class we considered. Note that $\mathrm{rk}(\varphi) = \mathrm{rk}(A) = 1$, so $\mathrm{null}\,\varphi = 2$.

We consider the sequence of subspaces $V = \mathrm{Ker}\,\varphi^2 \supset \mathrm{Ker}\,\varphi \supset \{0\}$. The first one relative to the second one is one-dimensional (since $\mathrm{null}\,\varphi^2 - \mathrm{null}\,\varphi = 3 - 2 = 1$).

Note that the kernel of φ has a basis consisting of the vectors $\begin{pmatrix} 1/3 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1/3 \\ 0 \\ 1 \end{pmatrix}$ (corresponding to the values $s = 1, t = 0$ and $s = 0, t = 1$ of the free variables respectively). The reduced column echelon form of the corresponding matrix $A = \begin{pmatrix} 1/3 & -1/3 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the matrix $R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -3 & 1 \end{pmatrix}$.

Since the basis vectors of $\mathrm{Ker}\,\varphi$ have pivots in the first and the second row, it is easy to see that the vector $f = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ can be taken as a basis of V relative to $\mathrm{Ker}\,\varphi$.

This vector gives rise to vector $\varphi(f) = \begin{pmatrix} -1 \\ -4 \\ -1 \end{pmatrix}$. It remains to find a basis of $\mathrm{Ker}\,\varphi$ relative to the span of $\varphi(f)$. Column reduction of the basis vectors of $\mathrm{Ker}(\varphi)$ by $\varphi(f)$ leaves us with the vector $g = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

Overall, $f, \varphi(f), g$ form a basis of V. The matrix of φ relative to this basis is $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Example 2. Now let $V = \mathbb{R}^3$, where φ is multiplication by the matrix $A = \begin{pmatrix} 21 & -7 & 8 \\ 60 & -20 & 23 \\ -3 & 1 & -1 \end{pmatrix}$.

In this case, φ^2 is multiplication by the matrix $\begin{pmatrix} -3 & 1 & -1 \\ -9 & 3 & -3 \\ 0 & 0 & 0 \end{pmatrix}$, $\varphi^3 = 0$, $\mathrm{rk}\,\varphi = 2$, $\mathrm{rk}\,\varphi^2 = 1$, $\mathrm{rk}\,\varphi^k = 0$ for $k \geqslant 3$, $\mathrm{null}(\varphi) = 1$, $\mathrm{null}(\varphi^2) = 2$, $\mathrm{null}(\varphi^k) = 3$ for $k \geqslant 3$.

We consider the sequence of subspaces $V = \mathrm{Ker}\,\varphi^3 \supset \mathrm{Ker}\,\varphi^2 \supset \mathrm{Ker}\,\varphi \supset \{0\}$. The first one relative to the second one is one-dimensional ($\mathrm{null}\,\varphi^3 - \mathrm{null}\,\varphi^2 = 1$).

The vector space $\mathrm{Ker}(\varphi^2)$ has a basis consisting of the vectors $\begin{pmatrix} 1/3 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1/3 \\ 0 \\ 1 \end{pmatrix}$ (corresponding to the values $s = 1, t = 0$ and $s = 0, t = 1$ of the free variables respectively). The reduced column echelon form of the corresponding matrix $A = \begin{pmatrix} 1/3 & -1/3 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the matrix $R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -3 & 1 \end{pmatrix}$.

Since the basis vectors of $\mathrm{Ker}\,\varphi^2$ have pivots in the first and the second row, the vector $f = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ can be taken as a basis of V relative to $\mathrm{Ker}\,\varphi^2$. This vector gives rise to the thread f, $\varphi(f) = \begin{pmatrix} 8 \\ 23 \\ -1 \end{pmatrix}$, $\varphi^2(f) = \begin{pmatrix} -1 \\ -3 \\ 0 \end{pmatrix}$. Since our space is 3-dimensional, this thread forms a basis. The matrix of φ relative to this basis is $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

Example 3. Let $V = \mathbb{R}^4$, where φ is multiplication by the matrix $A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & -1 \end{pmatrix}$.

In this case, $\varphi^2 = 0$, $\mathrm{rk}(\varphi) = 2$, $\mathrm{rk}(\varphi^k) = 0$ for $k \geqslant 2$, $\mathrm{null}(\varphi) = 2$, $\mathrm{null}(\varphi^k) = 4$ for $k \geqslant 2$.

We consider the sequence of subspaces $V = \mathrm{Ker}(\varphi^2) \supset \mathrm{Ker}(\varphi) \supset \{0\}$. The first one relative to the second one is two-dimensional ($\mathrm{null}\,\varphi^2 - \mathrm{null}\,\varphi = 2$).

The vector space $\mathrm{Ker}(\varphi)$ has a basis consisting of the vectors $\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ (corresponding to the values $s = 1, t = 0$ and $s = 0, t = 1$ of the free variables respectively). The reduced column echelon form has pivots in row one and row two, so the vectors $f_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $f_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ can be taken as a basis of V relative to Ker(φ).These vectors give rise to threads f_1, $\varphi(f_1) = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ and f_2, $\varphi(f_2) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$. These two threads together contain four vectors, so since our space is 4-dimensional, we have a basis. The matrix of φ relative to this basis is $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.