MA1112: Solutions to the midterm test

1. The intersection of these subspaces, by definition, is the set of all vectors of the form $a\begin{pmatrix}-1\\0\\1\end{pmatrix}+b\begin{pmatrix}3\\-2\\-1\end{pmatrix}$ whose coordinates satisfy the equation $x+y+4z=0$, which means that $(-a+3b)+(-2b)+4(a-b)=0$, or $3a-3b=0$, so $a=b$, and the intersection is spanned by the vector $\mathbf{h}=\begin{pmatrix}-1\\0\\1\end{pmatrix}+\begin{pmatrix}3\\-2\\-1\end{pmatrix}=\begin{pmatrix}2\\-2\\0\end{pmatrix}$. It is a one-dimensional space. Each of the subspaces U and V is two-dimensional: the first one is the span of two vectors that are not proportional, and the second one is the solution space to one linear equations in three unknowns, so there are two free variables. Thus, in each case, the relative basis consists of one vector, and it is enough to choose one vector in each subspace that is not proportional to $\mathbf{h}$. For the subspace U, we may take $\begin{pmatrix}-1\\0\\1\end{pmatrix}$. For the subspace V, we may take the basis vector corresponding to one of the free variables $\begin{pmatrix}-4\\0\\1\end{pmatrix}$. (The other basis vector is proportional to $\mathbf{h}$.)

2. Suppose that $\begin{pmatrix}x\\y\\z\end{pmatrix}$ is a vector in our plane. Its image under the given transformation is the vector $\begin{pmatrix}-26x+20y+21z\\11x-9y-9z\\-49x+40y+40z\end{pmatrix}$. A subspace is invariant if the image under every vector is again inside that same subspace, so we need to check if the coordinates of the result satisfy the same equation:

$$(-26x+20y+21z)-2(11x-9y-9z)-(-49x+40y+40z)=x-2y-z,$$

so the same equation is satisfied, and the subspace is invariant.

3. We begin with computing the characteristic polynomial of this matrix:

$$\det\begin{pmatrix}6-t & -23 & 14\\3 & -16-t & 10\\2 & -14 & 9-t\end{pmatrix}=-t^3-t^2+5t-3.$$

We note that $t=1$ is a root, which leads to a factorisation

$$-t^3-t^2+5t+3=-(t-1)^2(t+3),$$

so our linear transformation has eigenvalues 1 and -3. The multiplicity of the eigenvalue -3 is equal to one, so this eigenvalue has just one Jordan block of size 1. The multiplicity of the eigenvalue 1 is equal to two. Note that

$$A-I=\begin{pmatrix}5 & -23 & 14\\ 3 & -17 & 10\\ 2 & -14 & 8\end{pmatrix}$$

is a matrix of rank greater than 1 (since its rows are not all proportional), which tells us that out of two possible Jordan forms

$$\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -3\end{pmatrix}, \quad \begin{pmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 0 & 0 & -3\end{pmatrix},$$

the second one is correct: its rank after subtracting I is equal to two, and for the first one, the rank after subtracting I is equal to one. To find the Jordan basis, we need to compute the kernel of $(A-I)^2$. The answer is

$$(A-I)^2=\begin{pmatrix}-16 & 80 & -48\\ -16 & 80 & -48\\ -16 & 80 & -48\end{pmatrix}.$$

Note: we already know that the Jordan form is $\begin{pmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 0 & 0 & -3\end{pmatrix}$, and from this we can predict $\operatorname{rk}(A-I)^2=1$. Since we only care about $\ker(A-I)$, knowing one nonzero row of $(A-I)^2$ is enough: for a rank one matrix, they are all proportional. This can simplify computations. Now, let us find a Jordan basis. By a direct computation, the vector $\begin{pmatrix}-1/2\\ 1/2\\ 1\end{pmatrix}$ forms a basis for $\ker(A-I)$, and the vectors $\begin{pmatrix}5\\ 1\\ 0\end{pmatrix}$ and $\begin{pmatrix}-3\\ 0\\ 1\end{pmatrix}$ form a basis for $\ker(A-I)^2$. For a basis of $\ker(A-I)^2$ relatiive to $\ker(A-I)$, we may take any vector of the latter kernel not proportional to $\begin{pmatrix}-1/2\\ 1/2\\ 1\end{pmatrix}$, so for example $\mathbf{e}=\begin{pmatrix}5\\ 1\\ 0\end{pmatrix}$. We have $(A-I)\mathbf{e}=\begin{pmatrix}2\\ -2\\ -4\end{pmatrix}$. Finally, the vector $\mathbf{f}=\begin{pmatrix}1\\ 1\\ 1\end{pmatrix}$ forms a basis for $\ker(A+3I)$. Altogether, the vectors $\mathbf{e}, (A-I)\mathbf{e}, \mathbf{f}$ form a Jordan basis.

4. By rank–nullity theorem, we have $\operatorname{null}(\varphi^2)=4-\operatorname{rk}(\varphi^2)$, so $\operatorname{null}(\varphi^2)=2$. We know that $\ker(\varphi)\subseteq\ker(\varphi^2)$, so $\operatorname{null}(\varphi)=\dim\ker(\varphi)\leqslant\dim\ker(\varphi^2)=\operatorname{null}(\varphi^2)=2$, so the possible values of $\operatorname{null}(\varphi)$ are 0, 1, and 2. We note that the value 0 for nullity is impossible, since if $\operatorname{null}(\varphi) = 0$, then $\operatorname{null}(\varphi^2) = 0$: whenever $\varphi^2(v) = 0$, we have $\varphi(\varphi(v)) = 0$, implying $\varphi(v) = 0$ and implying $v = 0$; at the same time we know that $\operatorname{rk}(\varphi^2) = 2$, so $\operatorname{null}(\varphi^2) = 2$. The value 1 for nullity is possible; for example, it is attained for the linear transformation with the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The value 2 for nullity is also possible; for example, it is attained for the linear transformation with the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$