MA 1112: Linear Algebra II
Tutorial problems, February 12, 2019

**1.** The characteristic polynomial of $A$ is $t^2 - 2t + 1 = (t-1)^2$, so the only eigenvalue is $1$. We have $A - I = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}$, and this matrix evidently is of rank $1$. Also, $(A-I)^2 = 0$, so there is a stabilising sequence of subspaces $\text{Ker}(A-I) \subset \text{Ker}(A-I)^2 = V$. The dimension gap between these is equal to $1$, and we have to find a relative basis. The kernel of $A - I$ is spanned by the vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, and for the relative basis we can take the vector $f = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ which makes up for missing pivot. This vector gives rise to a thread $f, (A-I)f = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$, and reversing the order of vectors in the thread we get a Jordan basis $f, (A-I)f$, and the Jordan normal form $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

**2.** The characteristic polynomial of $A$ is $\det(A - tI) = -t^3 + 6t^2 - 12t + 8 = (2-t)^3$, so the only eigenvalue is equal to 2. Furthermore, $A - 2I = \begin{pmatrix} 4 & 5 & -2 \\ -8 & -10 & 4 \\ -12 & -15 & 6 \end{pmatrix}$, $(A-2I)^2 = 0$, $\text{rk}(A-2I) = 1$, $\text{rk}((A-2I)^2) = 0$. Thus, we have a sequence of subspaces $\text{Ker}(A-2I) \subset \text{Ker}(A-2I)^2 = \text{Ker}(A-2I)^3 = \ldots = V$. The kernel of $A - 2I$ is two-dimensional, and consists of all vectors $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ satisfying $4x + 5y - 2z = 0$, therefore $y$ and $z$ are free variables, and we have a basis of the kernel that consists of the vectors $\begin{pmatrix} -5/4 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1/2 \\ 0 \\ 1 \end{pmatrix}$. Bringing the matrix whose columns are these vectors to its reduced column echelon form, we observe that the missing pivot is the one in the third row, so the vector $e = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ forms a basis of $V$ relative to the kernel. We have $(A-2I)e = \begin{pmatrix} -2 \\ 4 \\ 6 \end{pmatrix}$. This vector belongs to the kernel, and we should find a basis of the kernel relative to the span of this vector. We reduce the basis vectors of the kernel using this vector:

$$\begin{pmatrix} -2 & -5/4 & 1/2 \\ 4 & 1 & 0 \\ 6 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} -2 & 0 & 0 \\ 4 & -3/2 & 1 \\ 6 & -15/4 & 5/2 \end{pmatrix},$$

and see that the both the second and the third column are proportional to the vector $f = \begin{pmatrix} 0 \\ 2 \\ 5 \end{pmatrix}$, which gives rise to a thread of length 1 and completes the basis. Overall, a Jordan basis is given by $e, (A - 2I)e, f$, and the Jordan normal form of our matrix is $\begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

**3.** The characteristic polynomial of $A$ is $\det(A - tI) = -t^3 - t^2 + t + 1 = (1-t)(1+t)^2$, so the eigenvalues are 1 and $-1$. Furthermore, $\text{rk}(A - I) = 2$, $\text{rk}(A - I)^2 = 2$, $\text{rk}(A + I) = 2$, $\text{rk}(A + I)^2 = 1$. Thus, the kernels of powers of $A - I$ stabilise instantly, so we should expect a thread of length 1 for the eigenvalue 1, whereas the kernels of powers of $A + I$ do not stabilise for at least two steps, so that would give a thread of length at least 2, hence a thread of length 2 because our space is 3-dimensional. To determine the basis of $\text{Ker}(A - I)$, we solve the system $(A - I)v = 0$ and obtain a vector $f = \begin{pmatrix} -6 \\ 4 \\ 1 \end{pmatrix}$. To deal with the eigenvalue $-1$, we see that the kernel of $A + I$ is spanned by the vector $\begin{pmatrix} -4 \\ 5/2 \\ 1 \end{pmatrix}$, the kernel of $(A + I)^2 = \begin{pmatrix} -24 & -48 & 24 \\ 16 & 32 & -16 \\ 4 & 8 & -4 \end{pmatrix}$ is spanned by the vectors $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. Reducing the latter vectors using the former one, we end up with the vector $e = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$, which gives rise to a thread $e, (A + I)e = \begin{pmatrix} 64 \\ 40 \\ -16 \end{pmatrix}$. Overall, a Jordan basis is given by $f, e, (A + I)e$, and the Jordan normal form of our matrix is $\begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.