MA 1112: Linear Algebra II
Tutorial problems, March 26, 2019

1. The characteristic polynomial of the matrix $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ is $2t - t^3 = t(2 - t^2)$, so the eigenvalues of this matrix are 0 and $\pm\sqrt{2}$. The matrix of the bilinear form corresponding to the quadratic form

$$q(x_1 e_1 + x_2 e_2 + x_3 e_3) = x_1 x_2 + x_2 x_3$$

is equal to $1/2$ of the matrix in question, so its signature can be read off the eigenvalues of this matrix, and is $(1, 1, 1)$.

2. We have

$$\frac{\partial \varphi}{\partial x_1} = 2\sin(x_1 - x_2)\cos(x_1 - x_2) - (x_1 + 2cx_2)e^{\frac{1}{2}(x_1^2 + x_2^2) + 2cx_1x_2} =$$
$$= \sin 2(x_1 - x_2) - (x_1 + 2cx_2)e^{\frac{1}{2}(x_1^2 + x_2^2) + 2cx_1x_2},$$

$$\frac{\partial \varphi}{\partial x_2} = -2\sin(x_1 - x_2)\cos(x_1 - x_2) - (2cx_1 + x_2)e^{\frac{1}{2}(x_1^2 + x_2^2) + 2cx_1x_2} =$$
$$= -\sin 2(x_1 - x_2) - (2cx_1 + x_2)e^{\frac{1}{2}(x_1^2 + x_2^2) + 2cx_1x_2},$$

and

$$\frac{\partial^2 \varphi}{\partial x_1^2} = 2\cos 2(x_1 - x_2) - e^{\frac{1}{2}(x_1^2 + x_2^2) + 2cx_1x_2} - (x_1 + 2cx_2)^2 e^{\frac{1}{2}(x_1^2 + x_2^2) + 2cx_1x_2},$$
$$\frac{\partial^2 \varphi}{\partial x_1 \partial x_2} = -2\cos 2(x_1 - x_2) - 2ce^{\frac{1}{2}(x_1^2 + x_2^2) + 2cx_1x_2} - (x_1 + 2cx_2)(2cx_1 + x_2)e^{\frac{1}{2}(x_1^2 + x_2^2) + 2cx_1x_2},$$
$$\frac{\partial^2 \varphi}{\partial x_2^2} = 2\cos 2(x_1 - x_2) - e^{\frac{1}{2}(x_1^2 + x_2^2) + 2cx_1x_2} - (2cx_1 + x_2)^2 e^{\frac{1}{2}(x_1^2 + x_2^2) + 2cx_1x_2},$$

so the matrix A is

$$\begin{pmatrix} 1 & -2 - 2c \\ -2 - 2c & 1 \end{pmatrix}.$$

By Sylvester's criterion, this quadratic form is positive definite if and only if $\Delta_2 = 1 - (2 + 2c)^2 > 0$ (since $\Delta_1 = 1$). We have

$$1 - (2 + 2c)^2 = (1 + 2 + 2c)(1 - 2 - 2c) = (3 + 2c)(-1 - 2c),$$

so the quadratic form is positive definite for $-3/2 < c < -1/2$. In particular, this holds for $c = -3/5$.

3. As we mentioned in solutions to an earlier homework, (A, A) is equal to the sum of squares of all matrix elements of A. The i, j-th matrix element of AB is $a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}$. The square of this number is, by Cauchy–Schwarz inequality, less than or equal to

$$(a_{i1}^2 + \ldots + a_{in}^2)(b_{1j}^2 + \ldots + b_{nj}^2),$$

the sum of squares of all elements in the i-th row of A times the sum of squares of all elements in the j-th column of B. Adding these up for all i and j, we get $\operatorname{tr}(AA^T)\operatorname{tr}(BB^T)$.

4. Let us prove a stronger result. Suppose $v_1, \dots, v_k$ are vectors that form pairwise obtuse angles. We shall demonstrate that if we throw away the vector v_k, the rest are linearly independent. Assume that the vectors $v_1, \dots, v_{k-1}$ are linearly dependent: $c_1 v_1 + \ldots + c_{k-1} v_{k-1} = 0$. Without the loss of generality, we may assume that $c_1, \ldots, c_l > 0$, and $c_{l+1}, \ldots, c_{k-1} < 0$ (the general case differs by re-numbering, and by suppressing the terms with zero coefficients). Obviously, $1 \leqslant l < k-1$: if all the coefficients are positive, or all the coefficients are negative, the scalar product

$$(e_k, c_1 e_1 + \ldots + c_{k-1} e_{k-1}) = c_1 (e_k, e_1) + \ldots + c_{k-1} (e_k, e_{k-1})$$

cannot be equal to 0. We have

$$0 \leqslant (c_1 e_1 + \ldots + c_l e_l, c_1 e_1 + \ldots + c_l e_l) = (c_1 e_1 + \ldots + c_l e_l, -c_{l+1} e_{l+1} - \ldots - c_{k-1} e_{k-1}) = \\ = \sum_{1 \leqslant i \leqslant l, l+1 \leqslant j \leqslant k-1} -c_i c_j (e_i, e_j),$$

which is clearly negative, a contradiction.