MA2215: Fields, rings, and modules
Homework problems due on November 12, 2012

Disclaimer: this assignment is for reading week, it is intentionally longer than all other ones since you will have more time. To make up for it, the mark for this assignment will carry the weight 1.5 in the overall mark for continuous assessment.

1. In class, we proved that for a principal ideal domain R, the maximal ideals are precisely the ideals generated by irreducible elements. Show that in the ring $\mathbb{Z}[t]$, which is, as we know, a UFD but not a PID, (**a**) the ideal generated by an irreducible element t^2+1 is not maximal; (**b**) the ideal $(t, 2) = \{tf(t) + 2g(t) : f(t), g(t) \in \mathbb{Z}[t]\}$ is maximal.

2. Explain why the field of fractions of the integral domain $\mathbb{Z}[i]$ is isomorphic to the quotient ring $\mathbb{Q}[t]/(t^2 + 1)\mathbb{Q}[t]$.

3. Show that a finite integral domain R is necessarily a field. (*Hint*: assume that we have $a \neq 0$ such that ab is different from 1 for all b. Then among the elements ax, where x runs over R, there will be two equal elements (why?).)

4. Show that in every field consisting of 4 elements we have $1 + 1 = 0$. (*Hint*: since this field, considered just with addition as an operation, is a group of order 4, we may only have $1 + 1 = 0$ or $1 + 1 + 1 + 1 = 0$ (why?). Also, $1 + 1 + 1 + 1 = (1 + 1)(1 + 1)$.)

5. Show that every field consisting of 4 elements is isomorphic to $\mathbb{F}_2[t]/(t^2 + t + 1)\mathbb{F}_2[t]$. (*Hint*: apart from 0 and 1, there is some element $a \neq 0, 1$, and the remaining element of the field is $1 + a$. What could a^2 be possibly equal to?)

6. Show that every field consisting of 3 elements is isomorphic to $\mathbb{F}_3$.

7. Show that in every field consisting of 9 elements we have $1 + 1 + 1 = 0$.

8. Show that every field consisting of 9 elements is isomorphic to $\mathbb{F}_3[t]/(t^2+1)\mathbb{F}_3[t]$. (*Hint*: apart from 0, 1 and $-1 = 1 + 1$, there is an element $a \neq 0, \pm 1$, so that the nine elements of the field are $0, \pm 1, \pm a, \pm 1 \pm a$. Therefore a^2 is equal to one of these elements. Show that for each of the possible variants, it is possible to express $\sqrt{-1}$ using a.)