MA2215: Fields, rings, and modules
Homework problems due on November 26, 2012

1. Show that $\mathbb{Q}(\sqrt{2}, \sqrt{3}) = \mathbb{Q}(\sqrt{2} + \sqrt{3})$.

2. Find a polynomial $f(x)$ of degree 4 with rational coefficients for which $f(\sqrt{2} + \sqrt{3}) = 0$. Show that $f(x)$ is irreducible in $\mathbb{Q}[x]$. (*Hint*: you know all the roots of this polynomial, so it must be easy to check how it can factorise.)

3. Deduce from the previous question that $[\mathbb{Q}(\sqrt{2} + \sqrt{3})\colon \mathbb{Q}] = 4$. Furthermore, explain, how this result implies that $\sqrt{3} \notin \mathbb{Q}(\sqrt{2})$.

4. In the field $\mathbb{F}_9 = \mathbb{F}_3(i)$, find an element b such that $b^8 = 1$ but $b^k \neq 1$ for $1 \leqslant k \leqslant 7$. Deduce that the group of invertible elements $\mathbb{F}_9^\times$ is a cyclic group of order 8.

5. Suppose that K is a field extension of F, and that $[K\colon F]$ is a prime number. Show that $K = F(\alpha)$ for some $\alpha \in F$. (*Hint*: any element of K which is not in F will do.)

6. Suppose that K is a field extension of F, and that $\alpha \in K$ is algebraic of odd degree over F. Show that $F(\alpha) = F(\alpha^2)$. (*Hint*: show that in general, $F(\alpha) = F(\alpha^2)$ or $[F(\alpha)\colon F(\alpha^2)] = 2$.)