MA2215: Fields, rings, and modules
Tutorial problems, November 15, 2012

In some of these questions, the *Eisenstein criterion* (formulated in class on Monday) could be useful. It states that, if R is a PID, $p \in R$ is an irreducible element, and $F = \mathrm{Frac}(R)$, then whenever for a polynomial $f(x) \in R[x]$ all the coefficients except for the leading term are divisible by p, the leading term is not divisible by p, and the constant term is not divisible by p^2, the polynomial $f(x)$ is irreducible in $F[x]$.

1. Show that for every n the polynomial $x^n - 175$ is irreducible in $\mathbb{Q}[x]$.

2. (**a**) Show that the polynomial $f(x) = x^2 + x + 4 \in \mathbb{Z}[x]$ does not satisfy the conditions of the Eisenstein criterion for any p, but after substituting $x + 2$ instead of x the Eisenstein criterion becomes applicable. Explain how to prove that $f(x)$ is irreducible.

(**b**) Show that not only $x^4 + 4$ does not satisfy the Eisenstein criterion for any p, but moreover, this polynomial is not irreducible.

3. Show that the polynomial $x^2 + y^3 - 1$ is irreducible in $\mathbb{C}[x, y]$. (*Hint*: we have $\mathbb{C}[x, y] \simeq (\mathbb{C}[x])[y]$: every polynomial in x and y is a polynomial in y whose coefficients are polynomials in x, and vice versa.)

4. Show that the polynomial $f(x) = 42x^4 - 8x^3 + 12x^2 - 6x - 1$ is irreducible in $\mathbb{Q}[x]$. (*Hint*: look at $f(1/x)$.)

Optional question (if you have some time left):

5. (**a**) Let $a_1, \ldots, a_n$ be distinct integers. Show that the polynomial

$$f(x) = (x - a_1)(x - a_2) \cdots (x - a_n) - 1$$

is irreducible over integers . (*Hint*: if $f = gh$ is a factorisation, look at the values of $g + h$ at $a_1, \ldots, a_n$.)

(**b**) Same question for the polynomial

$$f(x) = (x - a_1)^2(x - a_2)^2 \cdots (x - a_n)^2 + 1.$$