MA2316: Introduction to Number Theory
Tutorial problems for March 13, 2014

"Diophantine equations and irreducible polynomials"

1. Show thatif (a, b, c) is an integer solution to the equation $x^2 + y^2 + z^2 = 2xyz$, then all the three numbers a, b, c are even. Use it to deduce that this equation has the only solution $(0, 0, 0)$. (*Warning*: the triple $(a/2, b/2, c/2)$ is not a solution to the same equation!)

2. Show that if (a, b, c) is an integer solution to the equation $x^2 + y^2 + z^2 = xyz$, then all the three numbers a, b, c are divisible by 3. Use it to deduce that there is a one-to-one correspondence between solutions to that equation and solutions to the Markov's equation $x^2 + y^2 + z^2 = 3xyz$.

3. Using the identities $x^4+1 = x^4-(-1) = (x^2+1)^2-2x^2 = (x^2-1)^2-(-2x^2)$ (or otherwise) show that for each prime p the polynomial $x^4 + 1$ is reducible if viewed as a polynomial with coefficients in the field $\mathbb{Z}/p\mathbb{Z}$.

4. Show that the polynomial $x^4 + 1$ is irreducible in $\mathbb{Z}[x]$.

5. Show that if p is a prime number, and $\gcd(b, p) = 1$, then $x^n + px + bp^2$ is either irreducible in $\mathbb{Z}[x]$ or has an integer root.

6. Consider the polynomial $f(x) = 9x^n + 6(x^{n-1} + x^{n-2} + \cdots + x^2 + x) + 4$. Show that this polynomial is irreducible in $\mathbb{Z}[x]$.

7. Using Mason–Stothers Theorem, show that there are no non-constant polynomials $f(t), g(t) \in \mathbb{C}[t]$ satisfying the equation $f^3 - g^2 = 1$.