MA2316: Introduction to Number Theory
Tutorial problems for April 3, 2014

Numbers: rational, irrational, algebraic and transcendental

1. Show that $\log_2 9$ is an irrational number, and explain why there exist two irrational numbers α and β such that α^β is rational.

2. Show that $\sqrt{2}+\sqrt{3}$ is an algebraic number, and find its minimal polynomial.

3. Show that $e = \sum\limits_{n\geqslant 0} \frac{1}{n!}$ is not an algebraic number of degree 2. (*Hint*: if $ae^2+be+c=0$, then $ae+b+ce^{-1}=0$, where $e^{-1} = \sum\limits_{n\geqslant 0} \frac{(-1)^n}{n!}$.)

4. Show that the number $\sum\limits_{k\geqslant 0} \frac{2^{2^k}}{3^{k^k}}$ is transcendental.

5. Suppose that D is a positive integer which is not a perfect square. Show that for each $A > 2\sqrt{D}$ there exist only finitely many rational numbers $\frac{m}{n}$ satisfying the inequality

$$\left|\frac{m}{n}-\sqrt{D}\right| < \frac{1}{An^2}.$$

(*Hint*: $\frac{m}{n}+\sqrt{D} = \frac{m}{n}-\sqrt{D}+2\sqrt{D}$.)

6. Show that $\frac{1}{\pi}\sin^{-1}\left(\frac{3}{5}\right)$ is irrational. (*Hint*: show that if we assume the contrary, then the number $\frac{3}{5}+\frac{4}{5}i = \frac{2+i}{2-i}$ is a complex root of 1, and explain why $(2+i)^n = (2-i)^n$ cannot hold in $\mathbb{Z}[i]$ for $n > 0$.)