ASYMPTOTIC ANALYSIS OF ARITHMETIC FUNCTIONS (MA2316, ELEVENTH WEEK)

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Theorem 1. *We have*

$$M_\tau(n) = \ln n + O(1).$$

Proof. Note that

$$nM_\tau(n) = \sum_{k\le n}\tau(k) = \sum_{k\le n}\sum_{ab=n} 1 = \sum_{a,b\le n\colon ab\le n} 1 =$$
$$= \sum_{a\le\sqrt{n}}\sum_{b\le n/a} 1 + \sum_{a\le\sqrt{n}}\sum_{b\le n/a} 1 - \sum_{a\le\sqrt{n},b\le\sqrt{n}} 1 = 2\sum_{a\le\sqrt{n}}\left\lfloor\frac{n}{a}\right\rfloor - \lfloor\sqrt{n}\rfloor^2 =$$
$$= 2\sum_{a\le\sqrt{n}}\frac{n}{a} + O(\sqrt{n}) + O(n) = 2(n\ln\sqrt{n} + O(n)) + O(n) = n\ln n + O(n).$$

Here, we use two estimates for the sum of inverse integers:

$$\sum_{k=1}^{n}\frac{1}{k} \le 1 + \int_1^n \frac{dx}{x} = 1 + \ln n$$

and

$$\sum_{k=1}^{n}\frac{1}{k} \ge \int_1^{n+1} \frac{dx}{x} = \ln(n+1),$$

which together imply that

$$\sum_{k=1}^{n}\frac{1}{k} = \ln n + O(1).$$

Dividing $nM_\tau(n) = n\ln n + O(n)$ by n, we get the required statement. □

Theorem 2. *We have*

$$M_\phi(n) = \frac{3n}{\pi^2} + O(\ln n).$$

Proof. Let us examine the function $\Phi(n) = nM_\phi(n) = \sum\limits_{k\le n}\phi(k)$. Extend it to all nonnegative real numbers, putting $\Phi(x) = \Phi(\lfloor x\rfloor)$ when x is not an integer. Let us first show that

$$\sum_{d\ge 1}\Phi(x/d) = \frac{\lfloor x\rfloor\lfloor x+1\rfloor}{2}.$$

Indeed, the right hand side is equal to the number of pairs (m,n) with $0\le m < n\le x$. The number of such pairs with $\gcd(m,n) = d$ is equal to $\Phi(x/d)$, since factoring out d reduces counting these pairs to counting pairs $0\le m'\le n'\le x/d$, and for each $n'\le x/d$ the number of allowed m' is $\phi(n')$.

To make use of the formula we just proved, we shall invoke the following generalisation of Möbius inversion which you will prove in the next tutorial:

Suppose f, g are two functions with complex values defined on $[0, +\infty)$, and assume in addition that $\sum_{k,d\geq 1} |f(x/(kd))| < +\infty$ (for instance, that happens when $f(x) = 0$ for $x < 1$). Show that if

$$g(x) = \sum_{d\geq 1} f(x/d),$$

then we have

$$f(x) = \sum_{d\geq 1} \mu(d) g(x/d).$$

Thanks to this statement, we have

$$nM_\phi(n) = \frac{1}{2}\sum_{k\geq 1} \mu(k)\lfloor n/k\rfloor\lfloor 1 + n/k\rfloor.$$

Let us play around with this formula a little bit. Clearly,

$$\lfloor n/k\rfloor\lfloor 1 + n/k\rfloor = (n/k + O(1))(n/k + O(1)) = (n^2/k^2 + O(n/k)),$$

so

$$nM_\phi(n) = \Phi(n) = \frac{1}{2}\sum_{k=1}^{n} \mu(k)(n^2/k^2 + O(n/k)) =$$
$$= \frac{n^2}{2}\sum_{k\geq 1}\frac{\mu(k)}{k^2} - \frac{n^2}{2}\sum_{k>n}\frac{\mu(k)}{k^2} + n\sum_{k=1}^{n}(O(1/k)) =$$
$$= \frac{n^2}{2}\sum_{k\geq 1}\frac{\mu(k)}{k^2} + O(n) + O(n\ln n) = \frac{3n^2}{\pi^2} + O(n\ln n).$$

Here we use obvious estimates

$$\left|\sum_{k>n}\frac{\mu(k)}{k^2}\right| \leq \sum_{k>n}\frac{1}{k^2} < \sum_{k>n}\frac{1}{k(k-1)} = \frac{1}{n}$$

and

$$\sum_{k=1}^{n}\frac{1}{k} = O(\ln n)$$

which we proved above already, as well as the formula

$$\sum_{k\geq 1}\frac{\mu(k)}{k^2} = \prod_{p\text{ prime}}\left(1 - \frac{1}{p^2}\right) = \frac{1}{\prod\limits_{p\text{ prime}}\left(1 - \frac{1}{p^2}\right)^{-1}} = \frac{1}{\sum\limits_{m\geq 1}\frac{1}{m^2}} = \frac{1}{\frac{\pi^2}{6}} = \frac{6}{\pi^2}.$$

Dividing by n, we get

$$M_\phi(n) = \frac{3n}{\pi^2} + O(\ln n),$$

as required. □

This theorem easily implies that the "probability" for two randomly chosen numbers to be coprime is $\frac{6}{\pi^2}$. (If we agree that the probability in question is the (limit as $N \to \infty$ of the) proportion of pairs (m, n) with coprime m, n among all pairs m, n with $0 \leq m, n \leq N$).