MA2317: Introduction to Number Theory
Homework problems due October 15, 2010

1. (**a**) Compute gcd$(23171, 120959)$.
(**b**) Find some integers x and y such that

$$23171x + 120959y = \gcd(23171, 120959).$$

(**c**) Describe all pairs (x, y) satisfying the condition of the previous question.

2. (**a**) Prove that for the sequence of Fibonacci numbers $f_0 = 0$, $f_1 = 1$, $f_{n+2} = f_{n+1} + f_n$, we have $f_{n+k} = f_{n+1}f_k + f_nf_{k-1}$.
(**b**) Show that f_{mn} is divisible by f_n.
(**c**) Show that $\gcd(f_a, f_b) = f_{\gcd(a,b)}$.

3. The *least common multiple* of two integers a, b (notation: $\text{lcm}(a, b)$) is the smallest integer divisible by both a and b.
(**a**) Show that every common multiple of a and b is divisible by $\text{lcm}(a, b)$.
(**b**) Show that $\gcd(a, b) \cdot \text{lcm}(a, b) = ab$.

4. Define gcd for Gaussian integers (complex numbers with integer components) and compute $\gcd(15 + 25i, 36 - 2i)$.

5. (**a**) Modify the "$p_1p_2 \cdots p_n - 1$"-argument proving the infinitude of primes to show that there are infinitely many primes of the form $4k - 1$. (**b**) Why does not your proof adapt easily for primes of the form $5k - 1$?

6. Show that $\gcd(2^{2^n} + 1, 2^{2^m} + 1) = 1$ for $m \neq n$ (*Hint*: show that for $m > n$, $2^{2^m} - 1$ is divisible by $2^{2^n} + 1$), and use that fact to derive a yet another proof of the infinitude of primes.

7. Show that for the given $p_1, \ldots, p_m$ the number of integers of the form $p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}$ (for various choices of a_i) that do not exceed the given number n is at most

$$\left(1 + \frac{\ln n}{\ln p_1}\right)\left(1 + \frac{\ln n}{\ln p_2}\right) \cdots \left(1 + \frac{\ln n}{\ln p_m}\right),$$

and use that fact to derive a yet another proof of the infinitude of primes (*Hint*: $\lim\limits_{n\to\infty} \frac{(\ln n)^r}{n} = 0$ for every r).