MA2317: Introduction to Number Theory
Homework problems due December 3, 2010

The last two questions are optional. Other questions are assessed.

1. Show that $\sqrt{2}+\sqrt{3}+\sqrt{5}$ is an algebraic number.

2. Show that the number $\sum_{k\geqslant 0}\frac{2^{2^n}}{3^{n^n}}$ is transcendental.

3. Show that $e=\sum\limits_{n\geqslant 0}\frac{1}{n!}$ is not an algebraic number of degree 2, that is $ae^2+be+c=0$ with $a,b,c\in\mathbb{Z}$ implies $a=b=c=0$. (*Hint*: this can be rewritten as $ae+b+ce^{-1}=0$, where $e^{-1}=\sum\limits_{n\geqslant 0}\frac{(-1)^n}{n!}$.)

4. Consider the segment connecting the points $(0,3)$ and $(2n,0)$ in the plane. Explain what conditions should be satisfied by coefficients of a polynomial $f(x)$ in order for that segment to be the Newton diagram of that polynomial relative to the given prime p, and use these conditions to formulate a new criterion of irreducibility.

5. Find a rational parametrisation of the curve $y^2+3x^2-2x+y-1=0$.

6. Using the Mason–Stothers theorem, show that the equation

$$f(x)^n+g(x)^m+h(x)^k=0$$

for $2\leqslant n\leqslant m\leqslant k$ may have non-constant relatively prime solutions $f(x),g(x),h(x)\in\mathbb{C}[x]$ only for the following values of (n,m,k): $(2,2,k)$ (where $k\geqslant 2$), $(2,3,3)$, $(2,3,4)$, $(2,3,5)$.

7. Show that if for two (non necessarily relatively prime) polynomials $f(x)$ and $g(x)$ we have $h(x)=f(x)^3-g(x)^2\neq 0$, then $\deg f\leqslant 2\deg h-2$, $\deg g\leqslant 3\deg h-3$.