UNIVERSITY OF DUBLIN
TRINITY COLLEGE

Faculty of Science

school of mathematics

SF Mathematics **Trinity Term 2011**
JS Mathematics
JS TSM

Course 2317, a sample exam paper

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CREDIT WILL BE GIVEN FOR THE BEST **FOUR** QUESTIONS

For each task, the number of points you can get for a complete solution of that task is printed next to it.

You may use all statements proved in class and in home assignments; when using some statement, you should formulate it clearly, e.g. "in class, we proved that Euler's φ-function satisfies $\varphi(ab) = \varphi(a)\varphi(b)$ whenever a and b are coprime".

Non-programmable calculators are permitted for this examination.

1. (25 points) A number is said to be square free if no prime appears in its decomposition in degree greater than 1. Show that the sum
$$\sum_{\substack{n \text{ square free,} \\ n \le N}} \frac{1}{n}$$
tends to infinity as N tends to infinity, and use this result to obtain a yet another proof of the infinitude of primes.

2. (25 points) Compute the greatest common divisor $d(x)$ of polynomials $f(x) = x^5 - x^4 + x^2 - 3x - 2$ and $g(x) = 2x^4 - x^3 - 6x^2 + 2x + 3$, and find polynomials $p(x)$, $q(x)$ such that $d(x) = p(x)f(x) + q(x)g(x)$. (By "the" greatest common divisor, we mean the common divisor of maximal degree with leading coefficient 1.)

3. (a) (5 points) Show that every integer is congruent modulo 9 to the sum of its digits in base 10.

   (b) (5 points) Show that if $\gcd(a, 90) = 1$ then $a^{23} \equiv a^{-1} \pmod{90}$.

   (c) (15 points) Given that an integer n satisfies
$$n^{23} = 999356547346805156075552524294177648535563,$$
   explain how to find n without using a calculator/computer.

4. (25 points) Studying divisors of the values of the polynomial $16x^2 - 2$ at integer points, show that there are infinitely many primes $p \equiv -1 \pmod 8$.

5. (a) (10 points) Prove that there exist only finitely many rational numbers $\frac{m}{n}$ satisfying the inequality
$$\left|\sqrt{2} - \frac{m}{n}\right| < \frac{1}{n^3}.$$

   (b) (15 points) Find all rational numbers satisfying that inequality.

6. (a) (20 points) Show that the congruence $x^2 \equiv 1 \pmod{2^k}$ has one solution for $k = 1$, two solutions for $k = 2$, and four solutions for $k = 3$, and that for an odd prime p the congruence $x^2 \equiv 1 \pmod{p^k}$ always has exactly two solutions.

   (b) (5 points) How many solutions does the congruence $x^2 \equiv 1 \pmod{43120}$ have?