MA3413: Group representations I
Homework problems due on November 1, 2012

1. Let V be an n-dimensional vector space. Consider the representation of S_3 in the space $V \otimes V \otimes V$, where any $\sigma \in S_3$ permutes the factors accordingly:

$$\rho(\sigma)(v_1 \otimes v_2 \otimes v_3) = v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes v_{\sigma^{-1}(3)}.$$

(Note that we have to put σ^{-1} here to get a homomorphism: $\rho(\tau)\rho(\sigma)(v_1 \otimes v_2 \otimes v_3) = \rho(\tau)(v_{p_1} \otimes v_{p_2} \otimes v_{p_3})$, where $\sigma(p_i) = i$, and $\rho(\tau)(v_{p_1} \otimes v_{p_2} \otimes v_{p_3}) = v_{p_{q_1}} \otimes v_{p_{q_2}} \otimes v_{p_{q_3}})$, where $\tau(q_i) = i$. Therefore we have $\tau\sigma(p_{q_i}) = \tau(q_i) = i$, so $\rho(\tau)\rho(\sigma)(v_1 \otimes v_2 \otimes v_3) = \rho(\tau\sigma)(v_1 \otimes v_2 \otimes v_3)$.)

Compute the multiplicities of irreducible representations of S_3 in this representation.

We use the following notation for five irreducible representations representations of S_4: $\mathbb{1}$ — the trivial representation, sgn — the sign representation, U — the 2-dimensional representation (obtained from a nontrivial homomorphism $S_4 \to S_3$), V — the 3-dimensional *simplicial* representation (a summand of the permutation representation in $\mathbb{C}^4$), $V' \simeq V \otimes \text{sgn}$ — the other 3-dimensional representation.

2. Write down the character table of S_4, and decompose into irreducibles all pairwise tensor products of irreducible representations.

3. Do all irreducibles occur as constituents in tensor powers of a (not faithful) representation U of S_4?

Recall that the n^{th} symmetric power of a vector space W (which is denoted by $S^n(W)$) is a subspace in its n^{th} tensor power $W^{\otimes n}$ which is spanned by all symmetric products

$$w_1 \cdot w_2 \cdot \ldots \cdot w_n = \frac{1}{n!} \sum_{\sigma \in S_n} w_{\sigma(1)} \otimes w_{\sigma(2)} \otimes \ldots \otimes w_{\sigma(n)}$$

for all $w_1, \ldots, w_n \in W$. The n^{th} symmetric power $S^n(A)$ of an operator $A\colon W \to W$ is defined by

$$S^n(A)(w_1 \cdot w_2 \cdot \ldots \cdot w_n) = (Aw_1) \cdot (Aw_2) \cdot \ldots \cdot (Aw_n).$$

If (W, ρ) is a representation of a finite group G, $S^n(W)$ is an invariant subspace of all the operators $S^n(\rho(g))$ acting on $W^{\otimes n}$; this subspace is called the n^{th} symmetric power of the representation W.

4. Prove that $\chi_{S^2(V)}(g) = \frac{1}{2}(\chi_V(g)^2 + \chi_V(g^2))$. (*Hint*: recall that each individual matrix $\rho_V(g)$ can be diagonalised, use a basis of eigenvectors for V.)

5. Compute multiplicities of irreducibles in the following representations of S_4: (**a**) $S^2(V)$; (**b**) $S^2(V')$; (**c**) $S^2(U)$.

6. Let (V, ρ) be a complex representation of a finite group G.

(**a**) Prove that $G_\rho = \{g \in G \mid \rho(g) = \lambda \mathrm{Id}_V \text{ for some } \lambda \in \mathbb{C}\}$ is a normal subgroup of G.

(**b**) Prove that for any $g \in G$ we have $|\chi_V(g)| \leqslant \dim(V)$. (*Hint*: trace of an operator is equal to the sum of its eigenvalues.)

(**c**) Prove that $|\chi_V(g)| = \dim(V)$ if and only if g belongs to the subgroup G_ρ from the previous problem.

Optional question (does not count towards the continuous assessment): Prove *Hermite's reciprocity*: for the irreducible 2-dimensional representation V of S_3, and all positive integers k and l, we have $S^k(S^l(V)) \simeq S^l(S^k(V))$.