

MA3413: Group representations I
Homework problems due on November 15, 2012

Disclaimer: this assignment is for reading week, it is intentionally a bit longer than all other ones since you will have more time. To make up for it, the mark for this assignment will carry the double weight in the overall mark for continuous assessment.

1. Consider the action of S_4 on each of the following sets:

- (a) set of vertices of the tetrahedron;
- (b) set of edges of the tetrahedron;
- (c) set of faces of the tetrahedron;
- (d) set of vertices of the cube;
- (e) set of edges of the cube.

(In class, we considered the action on the set of faces of the cube.)

Each of these actions leads to a representation of S_4 on the vector space of functions on the corresponding finite set. Compute the characters of these representations, find multiplicities of irreducibles in them, and describe the corresponding invariant subspaces where those irreducibles are realised.

2. Let G be a group, and H a subgroup G such that $(G:H) = 2$.

(a) Show that H is normal.

(b) Show that for a conjugacy class of G which belongs to H one of two things can happen: (1) that conjugacy class is a single conjugacy class of H , so that two elements of H are conjugate in H iff they are conjugate in G or (2) that conjugacy class is a union of two conjugacy classes of H , which consist of the same number of elements.

(c) Show that for an irreducible representation of G one of two things can happen: (1) that representation remains irreducible when considered as a representation of H ; (2) as a representation of H it is isomorphic to a direct sum of two non-isomorphic irreducible representations. (*Hint:* the easiest is to use characters.)

3. (a) Describe conjugacy classes of A_4 .

(b) Explain what happens to each of irreducible representations of S_4 when restricted to A_4 .

(c) Describe all complex irreducible representations of A_4 .

4. (a) Describe conjugacy classes of A_5 .

(b) Show that the standard 4-dimensional irreducible representation V of S_5 (vectors in \mathbb{C}^5 with zero sum of coordinates) remains irreducible as a representation of A_5 .

(c) How many irreducible representations of A_5 will there be if we decompose the symmetric square $S^2(V)$ into irreducibles? Is the trivial representation a constituent of $S^2(V)$? Is V a constituent of $S^2(V)$?