MA3413: Group representations
Lecturer: Prof. Vladimir Dotsenko
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[1]Transcribed by Stiofáin Fordham. Last updated 16:19 Monday 21$^{\text{st}}$ October, 2013

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Lecture 1

24/9 12pm

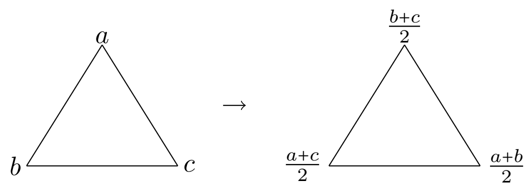
1.1. Introduction. To a large extent, the theory of group representations is an extension of linear algebra. Suppose we have a vector space without any additional structure and we are interested in classification up to isomorphism - then it is just the dimension n of the vector space.

Object	Classification
(vector space, subspace)	(n, m) with $n \geq m$ (where n is the dim. of the vector space, and m of the subspace)
(vector space, linear transformation)	dimension and Jordan normal form

The problem at hand about vector spaces is we want to identify the simplest "building blocks" and to explain how to build everything from those.

Some examples of potential applications of the theory

- Take a triangle, label the edges a, b, c. After some fixed time step, we replace the label at each vertex by the average, i.e. $\frac{b+c}{2}, \frac{a+c}{2}, \frac{a+b}{2}$ respectively. The question is then, if we repeatedly do this, will the values somehow stabilise after a long time? This can be solved using basic analysis but it is difficult. Instead we consider the generalised case of the n-polygon with rotational symmetry. It turns out that applying this procedure adds additional symmetry that, when investigated, gives an easy solution.



- Take a finite group G and write out its Cayley table. For example $\mathbb{Z}/3\mathbb{Z}$

	$\bar{0}$	$\bar{1}$	$\bar{2}$
$\bar{0}$	$\bar{0}$	$\bar{1}$	$\bar{2}$
$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{0}$
$\bar{2}$	$\bar{2}$	$\bar{0}$	$\bar{1}$

or for $\mathbb{Z}/2\mathbb{Z}$

	$\bar{0}$	$\bar{1}$
$\bar{0}$	$\bar{0}$	$\bar{1}$
$\bar{1}$	$\bar{1}$	$\bar{0}$

Now for each $g \in G$, introduce the variable x_g. For example in the case $\mathbb{Z}/2\mathbb{Z}$

$$\begin{pmatrix} x_{\bar{0}} & x_{\bar{1}} \\ x_{\bar{1}} & x_{\bar{0}} \end{pmatrix}$$

and investigate the determinant. Dedekind observed that the determinant seemed to factor very nicely in general. In the above case

$$\det\begin{pmatrix} x_{\bar{0}} & x_{\bar{1}} \\ x_{\bar{1}} & x_{\bar{0}} \end{pmatrix} = x_{\bar{0}}^2 - x_{\bar{1}}^2 = (x_{\bar{0}} + x_{\bar{1}})(x_{\bar{0}} - x_{\bar{1}})$$

He sent a letter to Frobenius describing this. The latter then tried to explain this and in doing so essentially founded the area of representation theory.

- We denote the symmetric group on n objects by S_n. Consider the cross-product, it satisfies

$$(x_1 \times x_2) \times x_3 + (x_2 \times x_3) \times x_1 + (x_3 \times x_1) \times x_2 = 0$$

This identity is invariant under permutation of indices (up to, perhaps, multiplication by -1). Objects like this, that do not change "much" under permutation of objects are studied extensively - they have significance in physics also. We would like to be able to classify these types of symmetries. Say we have a function of two arguments $f(x_1, x_2)$. We call functions that satisfy $f(x_1, x_2) = f(x_2, x_1)$ **symmetric**. Functions that satisfy $f(x_1, x_2) = -f(x_2, x_1)$ are called **skew-symmetric**. It is fairly easy to proove that every function of two arguments can be represented as the sum of a skew-symmetric function and a symmetric function. But for more than two variables the problem becomes much more difficult.

The prerequisites for this course

(1) Linear algebra: trace, determinant, change of basis, diagonalisation of matrices, vector spaces over arbitrary fields.
(2) Algebra: groups (for the most part up to order 8)[1], conjugacy classes & the class formula

$$|G| = \sum_{\text{conj. class } c} \frac{|G|}{|Z(c)|}$$

where $Z(c)$ is the centraliser of any element of the conjugacy class c. So in the case of $g_0 \in c$

$$Z(g_0) = \{g \in G \mid g_0 g = g g_0\}$$

Lecture 2

24/9 4pm

2.1. Examples of representations. As a first approximation, a representation tries to understand all about a group by looking at it as a vector space.

Definition 1. Let G be a group, and V a vector space (over a field k), then a **representation** of G on V is a group homomorphism

$$\rho: G \to \mathsf{GL}(V)$$

where $\mathsf{GL}(V)$ is the group of invertible linear transformations of the space V.

Examples

(1) Let G be arbitrary, and let $V = k$. We want a group homomorphism $G \to k^\times = k\backslash\{0\}$. The trivial representation is $\rho(g) = 1$ for every $g \in G$.

[1] Up to isomorphism: order 2 is $\mathbb{Z}/2\mathbb{Z}$, odd primes are $\mathbb{Z}/n\mathbb{Z}$ by the corollary to Lagrange's theorem, order 4 are the Kleinsche-vierergruppe $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/4\mathbb{Z}$, order 6 is $\mathbb{Z}/6\mathbb{Z}$ and $S_3 \cong D_3$ (the dihedral group: the symmetries of a regular n-gon) and for order 8 are, in the Abelian case, $\mathbb{Z}/8\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and in the non-Abelian case D_4 and the quaternions Q_8.

(2) (a) Let G be arbitrary, and let $V = kG :=$ the vector space with a basis $\{e_g\}_{g\in G}$. It is enough to understand how the element acts on each basis element $\rho_l(g)e_h := e_{gh}$. We must check that $\rho_l(g_1g_2) = \rho_l(g_1)\rho_l(g_2)$; it is enough to determine the effect of these elements on basis elements. We have

$$\begin{aligned}\rho_l(g_1g_2)e_h &= e_{(g_1g_2)h} = e_{g_1(g_2h)}\\ &= \rho_l(g_1)e_{g_2h} = \rho_l(g_1)\rho_l(g_2)e_h\end{aligned}$$

So $\rho_l(g_1g_2)$ and $\rho_l(g_1)\rho_l(g_2)$ map every basis vector to the same vector, hence are equal. This representation is called the left regular representation of G.

(b) G is arbitrary, $V = kG$. Let $\rho_r(g)e_h = e_{hg^{-1}}$ (*Exercise: show that defining this the other way around does not work.*). We have

$$\begin{aligned}\rho_r(g_1g_2)e_h &= e_{h(g_1g_2)^{-1}} = e_{hg_2^{-1}g_1^{-1}} = \rho_r(g_1)e_{hg_2^{-1}}\\ &= \rho_r(g_1)\rho_r(g_2)e_h\end{aligned}$$

(3) Let $G = \mathbb{Z}/2\mathbb{Z}$ and $V = k$. The only constraint is $\rho(\bar{1})^2 = 1$, so two different one-dimensional representations for $\operatorname{char} k \neq 2$ (there is only one one-dimensional representation for $\operatorname{char} k = 2$). We have

$$\begin{aligned}\rho(\bar{0}) &= 1\\ \rho(\bar{1}) &= 1\end{aligned}$$

(4) Let $G = \mathbb{Z}/2\mathbb{Z}$ and $V = k^2$. We have

$$\rho(\bar{0}) = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \qquad \rho(\bar{1}) = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

or

$$\rho(\bar{0}) = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \qquad \rho(\bar{1}) = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

(5) Take $G = S_3$ and $V = \mathbb{R}^2$. Take an arbitrary triangle in the plane, let the origin be the center of the triangle. Rotations of the triangle are represented by

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}\cos(\frac{2\pi}{3}) & -\sin(\frac{2\pi}{3})\\ \sin(\frac{2\pi}{3}) & \cos(\frac{2\pi}{3})\end{pmatrix}, \begin{pmatrix}\cos(\frac{4\pi}{3}) & -\sin(\frac{4\pi}{3})\\ \sin(\frac{4\pi}{3}) & \cos(\frac{4\pi}{3})\end{pmatrix}$$

Reflections are also linear transformations, so represented by matrices accordingly.

Definition 2. Let (V_1, ρ_1) and (V_2, ρ_2) be two representations of the same group G. Then their direct sum $(V_1 \oplus V_2, \rho_1 \oplus \rho_2)$ is defined as follows: the vectors of the representation are $V_1 \oplus V_2$, the homomorphisms $G \to \mathsf{GL}(V)(V_1 \oplus V_2)$ are defined as

$$(\rho_1 \oplus \rho_2)(g) = \begin{pmatrix}\rho_1(g) & 0\\ 0 & \rho_2(g)\end{pmatrix}$$

We check

$$\begin{aligned}(\rho_1 \oplus \rho_2)(g_1g_2) &= \begin{pmatrix}\rho_1(g_1g_2) & 0\\ 0 & \rho_2(g_1g_2)\end{pmatrix}\\ &= \begin{pmatrix}\rho_1(g_1)\rho_1(g_2) & 0\\ 0 & \rho_2(g_1)\rho_2(g_2)\end{pmatrix}\\ &= \begin{pmatrix}\rho_1(g_1) & 0\\ 0 & \rho_2(g_1)\end{pmatrix}\begin{pmatrix}\rho_1(g_2) & 0\\ 0 & \rho_2(g_2)\end{pmatrix}\end{aligned}$$

So it is a homomorphism.

Definition 3. Suppose that (V, ρ) is a representation of G. It is said to be **reducible** if there exists a subspace $U \subset V$ with $U \neq \{0\}$ or V, which is invariant with respect to all $\rho(g)$ i.e. $\rho(g)U \subset U$ for $g \in G$. Otherwise it is said to be irreducible.

The representation given above for S_3 is irreducible.

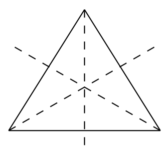
Lecture 3

27/9 1pm

When we stopped on Monday, we had given the definition of an irreducible representation and one example. We will now discuss some more examples.

Recall that the sum of two representations is just the direct sum of the vectors spaces - the block matrices, and an irreducible representation is a representation without a non-trivial invariant subspace. Aside: note that we will only discuss representations of positive dimension. $V_1 \oplus V_2$ has invariant subspaces V_1, V_2 so is not irreducible.

Last lecture I stated that the representation that I gave for S_3 constructed last time is irreducible. To see this: if it were not, all operators representing elements of S_3 would have a common eigenvector.



Over real numbers, each reflection has just two possible directions for eigenvectors so altogether no common eigenvectors. Over $\mathbb{C}$: real eigenvectors of reflections remain the only eigenvectors, so the same argument works.

3.1. Tensor products. Recall the tensor product: let V, W be vectors spaces, then $V \otimes W$ is the span of vectors (v, w) for $v \in V$ and $w \in W$ quotiented out by the relations

$$\begin{aligned} v \otimes (w_1 + w_2) &= v \otimes w_1 + v \otimes w_2 \\ (v_1 + v_2) \otimes w &= v_1 \otimes w + v_2 \otimes w \\ \lambda(v \otimes w) &= (\lambda v) \otimes w = v \otimes (\lambda w) \end{aligned}$$

to obtain the tensor product.

Definition 4. If (V, ρ) and (W, φ) are two representations of the same group G, then the **tensor product** $(V \otimes W, \rho \otimes \varphi)$ is defined as follows

$$(\rho \otimes \varphi)(g)(v \otimes w) := (\rho(g)(v)) \otimes (\varphi(g)(w))$$

extended to all $V \otimes W$ by linearity.

$\rho \otimes \varphi$ is a group homomorphism if ρ and φ are (it is extremely tedious to show that this is a homomorphism so we will not do it here).

Exercise: Let us fix a basis $\{e_i\}$ of V, a basis $\{f_j\}$ of W, and order a basis of $V \otimes W$ as follows

$$e_1 \otimes f_1, e_1 \otimes f_2, \dots, e_1 \otimes f_m, e_2 \otimes f_1, \dots, e_2 \otimes f_m, \dots, \dots, e_n \otimes f_1, \dots, e_n \otimes f_m$$

Suppose that $\rho(g)$ has the matrix A relative to $\{e_i\}$, $\varphi(g)$ has the matrix B relative to $\{f_j\}$

(1) Write down the matrix of $(\rho \otimes \varphi)(g)$ relative to the basis above.
(2) $\operatorname{tr}((\rho \otimes \varphi)(g)) = \operatorname{tr}(\rho(g)) \cdot \operatorname{tr}(\varphi(g))$.

Definition 5. If we have two representations (V_1, ρ_1) and (V_2, ρ_2) of the same group G, then they are said to be **isomorphic** or **equivalent** if there exists a vector space isomorphism $\varphi : V_1 \simeq V_2$ such that for each $g \in G$

$$\rho_2(g) \circ \varphi = \varphi \circ \rho_1(g)$$

$$\begin{tikzcd} V_1 \arrow[r, "\rho_1(g)"] \arrow[d, "\varphi"'] & V_1 \arrow[d, "\varphi"] \\ V_2 \arrow[r, "\rho_2(g)"'] & V_2 \end{tikzcd}$$

In other words, if we work with matrices, two representations are equivalent if they can be identified by change of coordinates.

To illustrate this definition, let us go back to one of the representations that we discussed on Monday.

Proposition 1. *For any G, $(kG, \rho_l) \simeq (kG, \rho_r)$.*

Proof. Let us define $\varphi : kG \to kG$ as follows: $\varphi(e_g) = e_{g^{-1}}$. This is invertible because if you apply it twice you get the identity. Let us check that it agrees with what we have here: we must show that $\varphi \circ \rho_l(g) = \rho_r(g) \circ \varphi$. It is enough to check this for the basis

$$\varphi \circ \rho_l(g)(e_h) = \varphi(e_{gh}) = e_{(gh)^{-1}}$$
$$\rho_r(g) \circ \varphi(e_h) = \rho_r(g)(e_{h^{-1}}) = e_{h^{-1}g^{-1}}$$

because $(gh)^{-1} = h^{-1}g^{-1}$, the result follows. $\square$

We will now deal with the question: is (kG, ρ_l) irreducible? The answer is no. For the case where G is finite, consider

$$v = \sum_{g \in G} e_g$$

Then for each $h \in G$,

$$\rho_l(h)(v) = v$$

because

$$\rho_l(h)\left(\sum_{g \in G} e_g\right) = \sum_{g \in G} e_{hg} = \sum_{g \in G} e_g = v$$

Now regarding textbooks: there is one by Curtis and Reiner "Methods of Representation Theory: With Applications to Finite Groups and Orders"; very comprehensive - ten times more than we could cover. Another is "Linear Representations of Finite Groups" by J.P. Serre - very concise, and clear but covers more than we can cover. And of course the textbook by Fulton and Harris.

Lecture 4

1/10 12pm

4.1. Decomposing the left-regular representation. The last thing we discussed on Thursday was the equivalence of representations which is just the same sort of equivalence that you know from linear algebra, it is just equivalence under change of coordinates essentially. There was an example over 1-dimensional subspaces of the left regular representation which looked exactly like the trivial representation. Let us look at that again because we will use this again often.

Let G be a finite group, and consider again kG equipped with the left regular representation ρ_l and let $v \in kG$. Consider

$$v = \sum_{g \in G} e_g$$

For each $g \in G$ we have $\rho_l(g)v = v$. The linear span of $kv \subset kG$ is a 1-dimensional subspace, invariant under all $\rho_l(g)$ and as a representation is isomorphic to the trivial representation.

Recall that our ultimate goal is to understand all representations of a finite group by looking at the irreducible representations. We now want to do this with this representation. We make some assumptions: either char $k = 0$ or char k is not a divisor of the number of elements in our group. Then there exists a subspace $W \subset kG$ which is invariant under all $\rho_l(g)$ such that $kG \simeq kv \oplus W$ - an isomorphism of representations, where we have

$$\rho_l \simeq \text{trivial} \oplus \rho_l\big|_W$$

Let us proove this

PROOF. Define $W := \{\sum_{g \in G} c_g e_g : \sum_{g \in G} c_g = 0\}$ which is invariant under all $\rho_l(g)$

$$\begin{aligned} \rho_l(h)\left(\sum c_g e_g\right) &= \sum c_g e_{hg} \\ &= \sum_{g \in G} c_{h^{-1}hg} e_{hg} \\ &= \sum c_{h^{-1}g'} e_{g'} \end{aligned}$$

with $\sum_{g' \in G} c_{h^{-1}g'} = 0$. There are now many ways to proceed. We have $\dim kv = 1$ and $\dim W = |G| - 1$ so it is enough to show that their intersection is equal to zero i.e. $kv \cap W = \{0\}$. For $\sum_{g \in G} c_g e_g \in kv$, all c_g are equal to each other. Take $g_0 \in G$ then

$$\sum_{g \in G} c_g = |G| \cdot c_{g_0}$$

If $\sum c_g e_g \in W$ we conclude that $|G| \cdot c_{g_0} = 0$ since in W all coefficients are equal to zero. If $c_{g_0} = 0$, then all $c_g = 0$, and we proved that $kv \cap W = \{0\}$. Otherwise $|G| = 0$ in k which is a contradiction. □

There is an alternative argument going in the other direction. We used the fact that if you have two subspaces of complementary dimensions, then their direct sum is the entire space iff they have zero intersection. Take some typical vector in your vector space kG

$$\sum_{g \in G} a_g e_g \in kG$$

then their sum can be written

$$\sum_{g \in G}\left(a_g - \frac{1}{|G|}\sum_{h \in G} a_h\right)e_g + \sum_{g \in G}\left(\frac{1}{|G|}\sum_{h \in G} a_h\right)e_g$$

where the "average" element given here forces: the first element belongs to W and the second one belongs to $k \cdot v$ so we have $kG = W + kv$[2] and since the dimensions are complementary, the sum must be direct.

One example that shows that our assumptions about characteristics are important: take $G = \mathbb{Z}/2\mathbb{Z}$ and $k = \mathbb{F}_2$. So G consists of the elements $\{\bar{0}, \bar{1}\}$ and $\mathbb{F}_2$ consists of $\{0, 1\}$. kG is a 2-dimensional representation of G over k. Then $v = e_{\bar{0}} + e_{\bar{1}}$ which is an invariant subspace. What happens now is that there is no way to choose a complementary subspace. In fact, *the only* invariant subspace is the one just given. How to see this? Any invariant subspace must be one-dimensional, suppose $ae_{\bar{0}} + be_{\bar{1}}$ spans it. Let us take this element under the action

$$\rho_l(\bar{1})(ae_{\bar{0}} + be_{\bar{1}}) = ae_{\bar{1}} + be_{\bar{0}}$$

So we must check under what conditions $ae_{\bar{1}} + be_{\bar{0}}$ is proportional to $ae_{\bar{0}} + be_{\bar{1}}$. The coefficient of proportionality is an element of k so therefore it is either 0 or 1. If it is 0 then $a = b = 0$ and it is not a 1-dimensional subspace, and if it is 1 then $a = b$ so it's $k \cdot v$. So we see in positive characteristic we get these undesirable things happening.

Let us formulate a more general result. From now on (except for rare explicit cases) we assume that $1/|G|$ make sense in in our field k - note this also assumes our group is finite.

Theorem 2. *Let (V, ρ) be a representation of G and let $U \subset V$ be invariant under all $\rho(g)$. Then there exists a subspace $W \subset V$ that is also invariant under all $\rho(g)$ and such that $V \simeq U \oplus W$ with*

$$\rho \simeq \rho\big|_U \oplus \rho\big|_W$$

Lecture 5

1/10 4pm

5.1. Complements of invariant subspaces. Continuing from last time the proof of the theorem. We will give two proofs. One will only work over $\mathbb{R}$ and with a modification over $\mathbb{C}$, whereas the other will work in generality.

PROOF #1: Take $k = \mathbb{R}$. Recall the proof in linear algebra that over real numbers all symmetric matrices are diagonalisable. Our proof will mirror this almost exactly.

Suppose that there exists a scalar product on our vector space which is invariant with respect to the group action

$$(\rho(g)(v), \rho(g)(w)) = (v, w)$$

i.e. they are all representable as orthogonal matrices. Then if $U \subset V$ is invariant, we shall show that $U^\perp$ is invariant. Indeed, let $v \in U^\perp$ and $u \in U$. We have

$$\begin{aligned}(\rho(g)(v), u) &= (\rho(g)(v), \rho(g)\rho(g)^{-1}(u)) \\ &= (v, \rho(g)^{-1}(u)) \\ &= (v, \rho(g^{-1})(u)) \\ &= 0\end{aligned}$$

[2]If we set $c_g = a_g - \frac{1}{|G|}\sum_{h\in G} a_h$ then $\sum_{g\in G} c_g = \sum_{g\in G}(a_g - \frac{1}{|G|}\sum_{h\in G} a_h) = \sum_{g\in G} a_g - \frac{|G|}{|G|}\sum_{h\in G} a_h = 0$ and the coefficients $\frac{1}{|G|}\sum_{h\in G} a_h$ clearly don't depend on $g \in G$.

because of $\rho(g^{-1})(u) \in U$ since U is invariant. It remains to show that an invariant scalar product always exists. How to show this? Take some scalar product $(v, w) \mapsto f(v, w)$ and define a new one

$$(v, w) = \frac{1}{|G|} \sum_{g \in G} f(\rho(g)(v), \rho(g)(w))$$

If we now apply some $\rho(h)$

$$\begin{aligned}(\rho(h)(v), \rho(h)(w)) &= \frac{1}{|G|} \sum_{g \in G} f(\rho(g)\rho(h)(v), \rho(g)\rho(h)(w)) \\ &= (v, w)\end{aligned}$$

□

This works on $\mathbb{R}$ because we have "positive definiteness" - a total ordering on $\mathbb{R}$. Otherwise we can have some non-trivial intersection of a subspace and its orthogonal complement: if we take $\mathbb{C}^2$ and $(z_1, w_1), (z_2, w_2) \mapsto z_1 z_2 + w_1 w_2$ then $(1, i)$ is orthogonal to itself.

PROOF #2: To define a direct sum decomposition on $V \simeq U \oplus W$ is equivalent to finding a linear operator P such that

- $P^2 = P$
- $\ker P = W$
- $\mathrm{Im}\ P = U$

> **Remark.** So how does it work? Suppose $P^2 = P$ then we want to show that $V = \ker P \oplus \mathrm{Im}\ P$. We want to check whether $\ker P \cup \mathrm{Im}\ P = \varnothing$. Let $v \in \ker P$. Then $Pv = 0$ and let $v = Pw$ then
>
> $$0 = Pv = PPw = P^2 w = Pw = v$$
>
> Then we write $v = (\mathrm{Id} - P)v + Pv$ and the first one is in the kernel and the second is in the image.

Moreover, if $P\rho(g) = \rho(g)P$ for all g, then the subspace $W = \ker P$ is invariant. Indeed, if $w \in W$ then we have

$$P\rho(g)(w) = \rho(g)P(w) = \rho(g)(0) = 0$$

which implies that $\rho(g)(w) \in \ker P = W$. Therefore, it is enough to show that there exists an operator P such that

(1) $P^2 = P$
(2) $U = \mathrm{Im}\ P$
(3) $\rho(g)P = P\rho(g)$ for all $g \in G$

The idea is to take some P_0 satisfying 1,2 - this is not difficult because this just means finding a decomposition on the level of the vector spaces. Now put

$$P \coloneqq \frac{1}{|G|} \sum_{g \in G} \rho(g) P_0 \rho(g)^{-1}$$

Condition three is satisfied (amounts to checking the same thing as in last lecture). But we must check that replacing P_0 by P does not "break" conditions one or two. So: why is $P^2 = P$? First we show that $\mathrm{Im}\ P \subset U$: if $v \in V$ then $\rho(g)^{-1}(v) \in V$ so $P_0\rho(g)^{-1}(v) \in U$ because for P_0, property two holds. Therefore $\rho(g)P_0\rho(g)^{-1}(v) \in U$ (due to invariance of U). So $P(v) \in U$. Next, suppose $u \in U$ then $\rho(g)^{-1}(u) \in U$ due to invariance of U. Therefore $P_0\rho(g)^{-1}(u) = \rho(g^{-1})(u)$ because of properties one and two for P_0[3]. Finally, $\rho(g)P_0\rho(g)^{-1}(u) = \rho(g)\rho(g^{-1})u = u$ so as a consequence $Pu = u$. Finally, finally, $P^2(v) = P(P(v)) = P(v)$ since $P(v) \in U$.

Finally, finally, finally, Im $P = U$ because $P(u) = u$ on U. □

Lecture 6

4/10

6.1. Intertwining operators and Schur's lemma. Last time we had: let V be a representation of G and $U \subset V$ an invariant subspace such that $V \simeq U \oplus W$. Our strategy was to take an operator P_0 such that $P_0 = P_0^2$ with Im $P_0 = U$. Then we average over G

$$P = \frac{1}{|G|} \sum_{g \in G} \rho(g) P_0 \rho(g)^{-1}$$

We showed that

- $P^2 = P$
- Im $P = U$
- $\rho(g)P = P\rho(g)$ for $g \in G$

Definition 6. Suppose that (V, ρ) is a representation of a group G, then operators T satisfying

$$\rho(g)T = T\rho(g) \quad \text{for all } g \in G$$

are called **intertwining** operators for V. More generally, if (V, ρ) and (W, φ) are two representations of the group G then $T : V \to W$ satisfying

$$T\rho(g) = \varphi(g)T$$

are called intertwining operators from V to W.

Theorem 3 (Schur's lemma)**.**

(1) *Suppose that (V_1, ρ_1) and (V_2, ρ_2) are irreducible representations of a group G. Then any intertwining operator $T : V_1 \to V_2$ is either zero or an isomorphism.*

(2) *Over the complex numbers (or any algebraically closed field), any intertwining operator $T : V_1 \to V_2$ is scalar.*

PROOF. Consider the kernel of T, it satisfies $\ker T \subset V_1$. Our claim is that it is invariant. Let $v \in \ker T$

$$T\rho_1(g)(v) = \rho_2(g)T(v) = \rho_2(g)(0) = 0$$

Therefore $v \in \ker T \Rightarrow \rho_1(g)(v) \in \ker T$. But it is irreducible. Therefore there are two cases: $\ker T = V_1$ or $\ker T = \{0\}$. If $\ker T = V_1$ then $T = 0$. If $\ker T = \{0\}$, T is injective. Why is T surjective? Well, let's consider the image, I claim that the image is an invariant subspace of V_2. Let's suppose that $w \in$ Im T and $w = T(v)$ then we have

$$\begin{aligned} \rho_2(g)(w) &= \rho_2(g)T(v) \\ &= T\rho_1(g)(v) \in \text{Im } T \end{aligned}$$

because T is injective. So either Im $T = 0$ or Im $T = V_2$. But Im $T = \{0\} \Rightarrow T = 0$. And Im $T = v_2 \Rightarrow T$ is surjective. So it is injective and surjective, thus an isomorphism as required.

For the second, part we will use the fact that every operator on a finite dimensional vector space has an eigenvalue. Assume that $T \neq 0$ so it is an isomorphism. Without loss of generality, $V_1 = V_2$ i.e. $T : V_1 \to V_1$. Let λ be an eigenvalue of T.

[3] If $P^2 = P$ and $U =$ Im P then $P =$ Id on U: let $v \in V$ and $v \neq 0$ and $Pv = w \neq 0$ say, then $P^2 v = Pv = w$ and $P(Pv) = Pw$ therefore $Pw = w$.

Then consider the operator $T - \lambda \cdot \mathrm{Id}$ which is an intertwining operator. It has a non-zero kernel

$$\ker(T - \lambda \cdot \mathrm{Id}) \neq \{0\}$$

and thus cannot be an isomorphism, so by the first part is zero, i.e. $T = \lambda \cdot \mathrm{Id}$. $\square$

We will see on Monday that this result allows us to build all the representation theory over complex numbers in a nice and simple way. For the next week and a half we will only look at representations where the ground field is the complex numbers, $k = \mathbb{C}$.

Informally, our goal is the following: since $\mathbb{C}$ has characteristic zero, our theorem from Monday about invariant complements holds. Our strategy is, if we have a complex representation then use this theorem, we will get two complementary subspaces which are invariant, then we apply it again and again until everything is irreducible. This will work since we are working over finite dimensional vector spaces.

Recall the left regular representation kG. Let us view this in the following way: this vector space has the structure of an algebra over k. The product is given on basis elements in the most straight-forward way as $e_g e_h := e_{gh}$. Associativity is a consequence of the associativity of the group operation. Then representations of a group G are in a one-to-one correspondence with left modules over this algebra kG - $kG \otimes M \to M$ or $kG \otimes kG \otimes M \rightrightarrows kG \otimes M$. kG has a bilinear form

$$\left(\sum_{g \in G} a_g e_g, \sum_{g \in G} b_g e_g \right) := \frac{1}{|G|} \sum_{g \in G} a_g b_{g^{-1}}$$

Theorem 4 (Orthogonality relations for matrix elements)**.** *Suppose that (V_1, ρ_1) and (V_2, ρ_2) are irreducible representations of a group G. Fix a basis $\{e_i\}_{i=1,\ldots,n}$ of V_1 and $\{f_i\}_{i=1,\ldots,m}$ of V_2. We have matrices $(\rho_1(g))_{i,j=,1\ldots,n}$ and $(\rho_2(g))_{i,j=,1\ldots,m}$ representing group elements. Fix i,j between 1 and n and k,l between 1 and m. We have $E^{(1)}_{ij}, E^{(2)}_{kl} \in kG$ with (where $E^{(1)}_{ij}(g)$ are the matrix elements of $\rho_1(g)$ with respect to the basis $\{e\}$ and vice versa $E^{(2)}_{kl}$ and $\rho_2(g)$ and $\{f\}$)*

$$E^{(1)}_{ij}(g) := \sum_{g \in G} \rho_1(g)_{ij} e_g$$

$$E^{(2)}_{kl}(g) := \sum_{g \in G} \rho_2(g)_{kl} e_g$$

Then

1. $(E^{(1)}_{ij}, E^{(2)}_{kl}) = 0$ *if* $V_1 \neq V_2$
2. $(E^{(1)}_{ij}, E^{(1)}_{kl}) = \frac{\delta_{jk}\delta_{il}}{\dim V_1}$

Lecture 7

8/10 12pm

We will proove the theorem stated at the end of the last class.

We want

$$\frac{1}{|G|} \sum_{g \in G} E^{(1)}_{ij}(g) E^{(2)}_{kl}(g^{-1}) = 0 \tag{7.1}$$

$$\frac{1}{|G|} \sum_{g \in G} E^{(1)}_{ij}(g) E^{(1)}_{kl}(g^{-1}) = \frac{\delta_{jk}\delta_{il}}{\dim V_1} = \begin{cases} 1/\dim V_1 & i = l,\ j = k \\ 0 & \text{otherwise} \end{cases} \tag{7.2}$$

PROOF. We want to proove the above identities for all $i, j < n$ and $k, l < m$. Let us take the operator $T_0 \colon V_1 \to V_2$ with $T_0(e_i) = f_l$ and $T_0(e_p) = 0$ for $p \neq i$.

We want to average over the elements so that the group action commutes, like an intertwining operator.

Define an operator T by

$$T = \frac{1}{|G|}\sum_{g\in G}\rho_2(g)^{-1}T_0\rho_1(g)$$

Recall, that this averaging operation has the following effect: the operator T now satisfies

$$\rho_2(g)T = T\rho_1(g) \quad \text{for all } g \in G$$

So T is an intertwining operator. Since V_1 and V_2 are non-isomorphic, by Schur's lemma, the operator must be zero. So now, let us write down this condition and try to extract some information from it. If this operator is written as a sum of basis elements then we get the zero matrix so we will have a lot of conditions. So we should apply T to a basis vector. Let's consider

$$\begin{aligned}\rho_1(g)(e_j) &= \sum_{k=1}^{n}\rho_1(g)_{kj}e_k\\ \Rightarrow T_0\rho_1(g)(e_j) &= \rho_1(g)_{ij}f_l\end{aligned}$$

because it is only non-zero on e_i from above. Now we apply the inverse

$$\begin{aligned}\rho_2(g)^{-1}T_0\rho_1(g)(e_j) &= \rho_2(g^{-1})(\rho_1(g)_{ij}f_l)\\ &= \rho_1(g)_{ij}\sum_{k=1}^{m}\rho_2(g^{-1})_{kl}f_k\end{aligned}$$

We considered the term $\rho_2(g)^{-1}T_0\rho_1(g)(e_j)$. Now we write, using the above

$$T(e_j) = \frac{1}{|G|}\sum_{g\in G}\sum_{k=1}^{m}E^{(1)}_{ij}(g)E^{(2)}_{kl}(g^{-1})f_k$$

This is a linear combination of f_k, so if it is equal to zero, then every coefficient must be zero. This gives us precisely the relations (7.1). We can get the second identity by adapting the above argument. First of all take $T_0(e_i) = e_l$ and $T_0(e_p) = 0$ for $p \neq i$ and

$$T = \frac{1}{|G|}\sum_{g\in G}\rho_1(g)^{-1}T_0\rho_1(g)$$

Our conclusion using Schur's lemma is not true now - we cannot conclude that it is zero. Instead it acts like a scalar. We need to see what kind of scalar we get, consider[4]

$$\begin{aligned}\operatorname{tr}(T) &= \frac{1}{|G|}\sum_{g\in G}\operatorname{tr}(\rho_1(g^{-1})T_0\rho_1(g))\\ &= \frac{1}{|G|}|G|\cdot\operatorname{tr}(T_0) = \delta_{il}\end{aligned}$$

Which is clear from the definition of T_0 given above. If T is a scalar λ then $\operatorname{tr}(T) = \lambda\cdot\dim V_2$ so we have

$$\lambda = \frac{\delta_{il}}{\dim V_1}$$

Similar to the above we have

$$\begin{aligned}\rho_1(g)^{-1}T_0\rho_1(g)(e_j) &= \rho_1(g^{-1})(\rho_1(g)_{ij}f_l)\\ &= \rho_1(g)_{ij}\sum_{k=1}^{m}\rho_1(g^{-1})_{kl}f_k\end{aligned}$$

[4] Use the cycle property of traces here $\operatorname{tr}(abc) = \operatorname{tr}(bca)$.

and eventually

$$\begin{aligned} T(e_j) &= \frac{1}{|G|} \sum_{g \in G} \sum_{k=1}^{m} E_{ij}^{(1)}(g) E_{kl}^{(1)}(g^{-1}) e_k \\ &= \lambda e_j = \frac{\delta_{il}}{\dim V_1} e_j \end{aligned}$$

which is actually equivalent to (7.2). □

Now we have one of the main definitions needed for the classification of irreducible representations of finite groups. In the next class we will discuss applications.

7.1. Characters.

Definition 7. For a representation (V, ρ) of G, we define the **character** χ_V as the function from G to the complex numbers $\chi_V : G \to \mathbb{C}$ given by

$$\chi_V(g) = \mathrm{tr}_V(\rho(g))$$

It is clear this does not depend on choice of basis, and only depends on the isomorphism classes of representations. Two basic properties that we will state before the class ends

(1) $\chi_{V_1} = \chi_{V_2}$ if $V_1 \simeq V_2$ which is obvious.
(2) $\chi(g^{-1}) = \overline{\chi(g)}$ for all $g \in G$.

Previously, we had the alternate proof of the decomposition into irreducibles which only worked for real or complex numbers, and we showed that there always exists an invariant Hermitian scalar product. The idea is that: if v is an eigenvector of $\rho(g)$, then

$$\begin{aligned} \lambda(v, v) = (\rho(g)(v), v) = (\rho(g)(v), \rho(g)\rho(g^{-1})(v)) &= (v, \rho(g^{-1})(v)) \\ &= (v, \lambda^{-1} v) \\ &= \overline{\lambda^{-1}}(v, v) \end{aligned}$$

So $\lambda = \overline{(\lambda^{-1})}$. So $\chi(g^{-1}) = \overline{\chi(g)}$.

Lecture 8

8/10 4pm

8.1. Orthogonality relations. The last thing was about characters: for every representation we have $\chi_V(g^{-1}) = \overline{\chi_V(g)}$. There is another reason why this should hold. Basically if you take the matrix $\rho(g)$, then because of one of the basic results of group theory, that is if you raise it to the power of the order of the group $\rho(g)^{|G|}$, then

$$\rho(g)^{|G|} = \rho(g^{|G|}) = \rho(1) = \mathrm{Id}$$

The conclusion is that for every element of a finite group, some power of the representative matrix is the identity, thus every representative matrix is diagonalisable. This can be seen by taking powers of the Jordan normal form - we can't have elements outside of the main diagonal otherwise. We have

$$\rho(g) \text{ is represented by } \begin{pmatrix} \chi_1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \chi_n \end{pmatrix} \text{ relative to some basis}$$

$$\rho(g^{-1}) \text{ is represented by } \begin{pmatrix} \chi_1^{-1} & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \chi_n^{-1} \end{pmatrix} = \begin{pmatrix} \overline{\chi_1} & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \overline{\chi_n} \end{pmatrix}$$

and the roots of unity all belong to the unit circle[5] $|z| = 1 \Leftrightarrow z\bar{z} = 1 \Leftrightarrow \bar{z} = z^{-1}$. In particular we have

$$\frac{1}{|G|}\sum_{g\in G}\chi_{V_1}(g)\chi_{V_2}(g^{-1}) = \frac{1}{|G|}\sum_{g\in G}\chi_{V_1}(g)\overline{\chi_{V_2}(g)}$$

Since we have

$$\chi_{V_1}(g) = \sum_{i=1}^{\dim V_1} E_{ii}^{(1)}(g)$$
$$\chi_{V_2}(g) = \sum_{j=1}^{\dim V_2} E_{jj}^{(2)}(g)$$

so we conclude that

$$\frac{1}{|G|}\sum_{g\in G}\chi_{V_1}(g)\chi_{V_2}(g^{-1}) = 0 \qquad \text{if } V_1 \text{ is not isomorphic to } V_2 \ (V_1, V_2 \text{ irreducible})$$
$$\frac{1}{|G|}\sum_{g\in G}\chi_{V_1}(g)\chi_{V_1}(g^{-1}) = 1 \qquad \text{for an irreducible representation } V_1$$

The first statement is easy because of orthogonality for matrix elements. The second:

$$\begin{aligned}\frac{1}{|G|}\sum_{g\in G}\chi_{V_1}(g)\chi_{V_1}(g^{-1}) &= \frac{1}{|G|}\sum_{g\in G}\sum_{j,k}E_{jj}(g)E_{kk}(g^{-1}) \quad (j = 1\dots\dim V_1,\ k = 1\dots\dim V_1) \\ &= \frac{1}{\dim V_1}\cdot\dim V_1 = 1\end{aligned}$$

Define for $\varphi, \psi{:}\, G \to \mathbb{C}$ the following

$$(\varphi, \psi) = \frac{1}{|G|}\sum_{g\in G}\varphi(g)\overline{\psi(g)}$$

Then if V_1, V_2 are irreducible representations, we have

$$(\chi_{V_1}, \chi_{V_2}) = \begin{cases} 1, & V_1 \simeq V_2 \\ 0, & V_1 \not\simeq V_2 \end{cases}$$

Clearly, if $V \simeq \underbrace{V_1 \oplus V_1 \oplus \dots \oplus V_1}_{a_1} \oplus \underbrace{V_2 \oplus V_2 \oplus \dots \oplus V_2}_{a_2} \oplus \dots \oplus \underbrace{V_n \oplus V_n \oplus \dots \oplus V_n}_{a_n}$ where we assume that the V_i are pairwise non-isomorphic irreducible representations, then we have

$$\chi_V = a_1\chi_1 + \dots + a_n\chi_n$$

[5]Since the group has finite order, the eigenvalues must have finite order, so they must be roots of unity etc.

and $(\chi_V, \chi_{V_i}) = a_i$. Furthermore, let us consider the representation $(V = \mathbb{C}G, \rho_l)$ (the left regular representation) and also W some irreducible representation. We will compute

$$(\chi_{\mathbb{C}G}, \chi_W)$$

The character of the left-regular representation is very easy to compute:

$$\rho_l(g)e_h = e_{gh}$$

If $g \neq 1$, all diagonal entries of $\rho_r(g)$ are zero[6], so $\chi_{\mathbb{C}G}(g) = 0$. The case of the identity element is easy since it goes to the identity matrix, so the value is just $\chi_{\mathbb{C}G}(1) = |G|$.

$$\begin{aligned}(\chi_{\mathbb{C}G}, \chi_W) &= \frac{1}{|G|}\sum_{g\in G}\chi_{\mathbb{C}G}\overline{\chi_W(g)} \\ &= \frac{1}{|G|}\chi_{\mathbb{C}G}(1)\overline{\chi_W(1)} = \dim W\end{aligned}$$

So we have proven something rather important here.

Theorem 5. *The decomposition of the left regular representation into a sum of irreducibles contains a copy of every irreducible representation with multiplicity equal to the dimension.*

This is a very powerful result, because up to now we had no reason to assume there were even finitely many non-isomorphic representations. Let us look at an example: take $G = S_3$, we have $|G| = 6$. Our theorem tells us there are no irreducible representations of dimension three - if there were then we would have three copies of a three dimensional space inside a six dimensional space. If G has two non-isomorphic two-dimensional irreducible representation say U_1, U_2, then $\mathbb{C}G$ contains $U_1 \oplus U_1 \oplus U_2 \oplus U_2$ so $\dim \geq 8$ a contradiction.

We know one two-dimensional representation U, so $\mathbb{C}G \simeq U \oplus U \oplus \underbrace{V_1 \oplus \cdots \oplus V_m}_{\text{1d. irreducible}}$ and $6 = 2 + 2 + m$ so $m = 2$, that is, there are two non-isomorphic irreducibles. The trivial representation is an example of a one-dimensional representation. Another is the sign representation - even permutations are mapped to 1 and odd permutations are mapped to -1 i.e. $\rho(\sigma) = \text{sgn}(\sigma)$. Then our claim is that these are all the irreducible representations of S_3 and hence we have classified all representations of S_3.

In conclusion, S_3 has three irreducible representations (up to isomorphism): the trivial representation, the sign representation and the representation by symmetries of a triangle.

In principle, the above theorem is enough to determine all irreducible representations, but we would prefer to have some more results.

One simple observation is that characters are invariant under conjugation

$$\chi_V(ghg^{-1}) = \chi_V(h) \qquad g, h \in G$$

This is simple because

$$\begin{aligned}\text{tr}_V(\rho(ghg^{-1})) &= \text{tr}_V(\rho(g)\rho(h)\rho(g)^{-1}) \\ &= \text{tr}_V(\rho(h)) = \chi_V\end{aligned}$$

In fact this gives us an upper bound on the number of irreducible representations: it is bounded by the number of conjugacy classes. We will proove this next time and show that this bound is sharp.

[6]It is clear from the way that $\rho_l(g)$ acts that the diagonal entries are either 1 or 0. If it was 1 then there would be some e_k that would get mapped to e_k, which is only possible if $g = 1$.

Lecture 9

11/10

9.1. The number of irreducible representations. Last time we discussed some things about characters of representations.

Theorem 6. *The number of non-isomorphic irreducible representations of a finite group G over $\mathbb{C}$ is equal to the number of conjugacy classes of G.*

PROOF. For any character χ_V of a representation V we have

$$\chi_V(ghg^{-1}) = \chi_V(h) \tag{9.3}$$

Functions that satisfy equation (9.3) are called **class functions**. We shall proove that characters of irreducibles form a basis in the space of class functions. Since the dimension of the space of class functions is clearly equal to the number of conjugacy classes, this will complete the proof. We have

$$\chi_V(ghg^{-1}) = \mathrm{tr}_V(\rho(ghg^{-1})) = \mathrm{tr}_V(\rho(g)\rho(h)\rho(g)^{-1}) = \mathrm{tr}_V(\rho(h)) = \chi_V(h)$$

It is harder to proove the part about bases. We will need to use the following lemma

Lemma 7. *Let (V,ρ) be a representation of G, $\varphi : G \to \mathbb{C}$ be a class function, then*

$$T_{\varphi,\rho} = \sum_{g\in G} \varphi(g)\rho(g)$$

is an intertwining operator.

PROOF.

$$\begin{aligned}
\rho(h)T_{\varphi,\rho}\rho(h)^{-1} &= \sum_{g\in G} \rho(h)\varphi(g)\rho(g)\rho(h)^{-1} \\
&= \sum_{g\in G} \varphi(g)\rho(hgh^{-1}) \\
&= \sum_{g\in G} \varphi(hgh^{-1})\rho(hgh^{-1}) \\
&= \sum_{g'\in G} \varphi(g')\rho(g') = T_{\varphi,\rho}
\end{aligned}$$

□

In particular, if V is irreducible, then by Schur's lemma, this operator $T_{\varphi,\rho}$ is a scalar λ so

$$\begin{aligned}
\dim V \cdot \lambda &= \mathrm{tr}(T_{\varphi,\rho}) \\
&= \sum_{g\in G} \varphi(g)\mathrm{tr}\rho(g) = \sum_{g\in G} \varphi(g)\chi_V(g) \\
&= |G| \cdot (\varphi, \bar{\chi}_V) \\
\Rightarrow \lambda &= \frac{|G|}{\dim V}(\varphi, \bar{\chi}_V)
\end{aligned}$$

Now, suppose that characters of irreducibles do not span the space of class functions. Then their complex conjugates do not span the space either. Therefore there exists a non-zero class function orthogonal to all $\bar{\chi}_V$ for irreducible V with respect to $(\,,)$. Denote it by ψ. We have $T_{\psi,\rho} = 0$ for irreducible representations (V,ρ). Because of our formula for $T_{\varphi,\rho}$ it follows that $T_{\psi,\rho} = 0$ for all representations (V,ρ).

Our objective is to show $T_{\psi,\rho}$ is zero to get a contradiction. Let us consider $(\mathbb{C}G, \rho_l)$, then

$$0 = T_{\psi,\rho_l} = \sum_{g\in G} \psi(g)\rho_l(g)$$

In particular we have

$$0 = T_{\psi,\rho}(e_1) = \sum_{g \in G} \psi(g)\rho_l(g)(e_1) = \sum_{g \in G} \psi(g)e_g$$

Therefore $\psi = 0$, because the e_g are linearly independent. Since characters of irreducibles are orthogonal they are linearly independent, therefore they form a basis of the space of class functions. □

Let us now summarise our knowledge for the specific case of S_3 which we discussed on Monday. We know that it has three irreducible representations. So it has three conjugacy classes: the three cycles, the two cycles (transpositions) and the identity. So let us write down the **character table** for S_3.

0	Id	(12),(13),(23)	(123),(132)
Trivial	1	1	1
Sign	1	-1	1
2d. U	2	0^7	$2\cos(\frac{2\pi}{3}) = -1$

$$(\chi_{\text{triv}}, \chi_{\text{sign}}) = \frac{1}{6}(1 \cdot 1 + 3 \cdot 1(-1) + 2 \cdot 1 \cdot 1)$$

The middle term above is

$$\chi_{\text{triv}}(12)\overline{\chi_{\text{sign}}(12)} + \chi_{\text{triv}}(13)\overline{\chi_{\text{sign}}(13)} + \chi_{\text{triv}}(23)\overline{\chi_{\text{sign}}(12)}$$

We also have

$$(\chi_{\text{triv}}, \chi_{\text{U}}) = \frac{1}{6}(1 \cdot 2 + 3 \cdot 1 \cdot 0 + 2 \cdot 1 \cdot (-1))$$

(these are weighted sums: the middle term is multiplied by three because there are three terms in that conjugacy class etc.) and

$$(\chi_{\text{U}}, \chi_{\text{U}}) = \frac{1}{6}(2 \cdot 2 + 3 \cdot 0 \cdot 0 + 2 \cdot (-1) \cdot (-1))$$

Lecture 10

15/10 12pm

The goal today is to show how characters can also be used to decompose a given representation into a direct sum of irreducible representations. We know this is true in principle, but we do not have a direct method for doing this.

Aside: suppose that V, W are two representations of a finite group G. As usual all our representations are over $\mathbb{C}$, unless stated otherwise. Then of course we know that they can be decomposed into direct sums of irreducibles

$$V \simeq \oplus_i V_i^{\oplus a_i}$$
$$W \simeq \oplus_i V_i^{\oplus b_i}$$

This is sort of like decomposing integers into primes. $\{V_i\}$ is a list of irreducible representations of G. But we have not shown that these numbers a_i, b_i are well-defined - that is, we have not shown that this decomposition is unique. The analogy is like the fundamental theorem of arithmetic, that decomposition into a prime factorisation is unique. To proove this, we use the orthogonality of characters

$$(\chi_V, \chi_{V_i}) = a_i$$
$$(\chi_{V_i}, \chi_{V_j}) = \delta_{ij} \qquad \text{(orthogonality)}$$

More generally, if we compute the scalar product of V, W

$$(\chi_V, \chi_W) = \sum_i a_i b_i$$

Our criterion for irreducibility is

$$V \text{ is irreducible} \Leftrightarrow (\chi_V, \chi_V) = 1$$

PROOF. We have $V \simeq \oplus V_i^{\oplus a_i}$ then $(\chi_V, \chi_V) = \sum_i a_i^2$. □

Now, given some arbitrary representation, it is difficult to determine whether a representation is irreducible by hand. But the above criterion makes it very easy. This alone might be enough to highlight the importance of characters.

It is possible to view the above formulas in a more conceptual way. We note that the characters are integers. There is a philosophy in mathematics called categorification: when we have integers and there's no particular reason to assume that they should all be integers, then we can interpret these integers as dimensions.

Theorem 8. *If $V \simeq \oplus V_i^{\oplus a_i}$ and $W \simeq \oplus V_i^{\oplus b_i}$ then*

$$\begin{aligned}(\chi_V, \chi_W) &= \sum_i a_i b_i \\ &= \dim \operatorname{Hom}_G(V, W)\end{aligned}$$

where $\operatorname{Hom}_G(V, W)$ is the space of intertwining operators between V and W.

Remark. This can be interpreted as an alternative way to view Schur's lemma.

PROOF. Linear operators between V and W are block $N \times N$ matrices

$$A = \begin{pmatrix} & A_{ij} & \\ & & \\ & & \end{pmatrix}$$

and A_{ij} is itself a block matrix

$$A_{ij} = \begin{pmatrix} & T_{pq}^{ij} & \\ & & \\ & & \end{pmatrix}$$

(where T_{pq}^{ij} is a matrix of a linear operator from the p^{th} copy of V_i to the q^{th} copy of V_j) and of course group elements $g \in G$ acting on V, W are represented by block diagonal matrices. The intertwining condition is

$$\rho_W(g) A = A \rho_V(g) \quad \text{for all } g \in G$$

Recall that the multiplication rules for block matrices imply that all T_{pq} are intertwining operators iff A represents an intertwining operator.

(Here is an example to illustrate this

$$\begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} r_1 a & r_2 b \\ r_1 c & r_2 d \end{pmatrix}$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix} = \begin{pmatrix} a s_1 & b s_2 \\ c s_1 & d s_2 \end{pmatrix}$$

and we equate these.)

Therefore A represents an intertwining operator(s) iff $T_{pq}^{ij} = 0$ for $i \neq j$ and T_{pq}^{ij} is a scalar whenever $i = j$ by Schur's lemma. A_{ii} depends on $a_i \times b_i$ parameters (because p, q range over a_i, b_i values respectively), so $\dim \operatorname{Hom}_G(V, W) = \sum_i a_i b_i$. □

Let us now describe how to decompose a representation into invariant subspaces using characters in a canonical fashion.

10.1. Canonical decomposition of a representation. Suppose $V \simeq U \oplus U$ where U is irreducible. We can say that the first factor contains all vectors of the form $(u, 0)$ and the second, all vectors $(0, u)$. Alternatively we can take $U_1 = \{(u, 0)\}$ and $U_2 = \{(u, u)\}$. These obviously have zero intersection, so U_1, U_2 are invariant, so $V = U_1 \oplus U_2$. So in general the direct sum decomposition cannot be unique. However, what is true is that if we fix some irreducible representation and take the direct sum of all invariant subspaces isomorphic to that space, then the decomposition will be well-defined. That is, if $V \simeq \oplus_i V_i^{\oplus a_i}$ then for each i, $V_i^{\oplus a_i}$ is a well-defined subspace, i.e. it does not depend on the choice of decomposition of V. What we shall proove in the next class is the following

Theorem 9. *Let us define (for V a representation of G)*

$$p_i = \frac{\dim V_i}{|G|} \cdot \sum_{g \in G} \overline{\chi_{V_i}(g)} \rho_V(g)$$

then $p_i^2 = p_i$, $p_i \in \mathrm{Hom}_G(V, V)$ and $\mathrm{Im}\ p_i = V_i^{\oplus a_i}$

Lecture 11

15/10 4pm

We will proove the theorem from the last class.

PROOF. First of all, let us proove that $p_i \in \mathrm{Hom}_G(V, V)$ i.e. that p_i is an intertwining operator. For $\overline{\chi}_i$ ($\chi_i := \chi_{V_i}$) then

$$T_{\overline{\chi}_i, \rho_V} = \sum_{g \in G} \overline{\chi}_i(g) \rho_V(g) \tag{11.4}$$

is an intertwining operator, which we prooved last week. If V is an irreducible representation, then

$$T_{\overline{\chi}_i, \rho_V} = \frac{|G|}{\dim V} (\chi_{V_i}, \chi_V)$$

So, it is equal to zero if $V_i \not\simeq V$ and equal to $|G| / \dim V$ if $V \simeq V_i$, under the assumption of irreducibility, but of course in general then (11.4) tells us that $T_{\overline{\chi}_i, \rho_V}$ is equal to $|G| / \dim V_i$ on each constituent V_i and equal to zero otherwise. Then

$$p_i = \frac{\dim V_i}{|G|} T_{\overline{\chi}_i, \rho_V}$$

is equal to 1 on each irreducible constituent isomorphic to V_i, and to zero on all constituent V_j for $j \neq i$. Now parts (1) to (3) follow. □

In particular, from this formula, we can extract a sum of all the copies of the trivial representation. If we consider the operator

$$p = \frac{1}{|G|} \sum_{g \in G} \rho_V(g)$$

is the projector on the subspace of invariant vectors in V ($\rho_V(g)v = v$).

11.1. Further decomposition into irreducibles. Our set-up is that $V = \oplus_i V^{(i)}$ with $V^{(i)} \simeq V_i^{\oplus a_i}$. So $V^{(i)} = \mathrm{Im}\ p_i$ constructed in the previous theorem. Now let us assume that we "know" irreducibles V_i such that we know the matrices $\rho_i(g) = (E_{kl}^{(i)}(g))$ where $k, l = 1, \dots, n_i$ where $n_i = \dim V_i$. Now fix one of the i and define the operators

$$P_{kl}^{(i)} = \frac{\dim V_i}{|G|} \sum_{g \in G} E_{lk}^{(i)}(g^{-1}) \rho_V(g)$$

(note the form of the matrix element in the sum - it is the transpose inverse)

Theorem 10. (1) *$P_{kk}^{(i)}$ are projectors, $(P_{kk}^{(i)})^2 = P_{kk}^{(i)}$ (but not intertwining operators). The subspaces $V^{(i,k)} := \mathrm{Im}\ P_{kk}^{(i)} \subset V^{(i)}$ for $k = 1, \dots, n_i$ and $V^{(i)} = \oplus_{k=1}^{m_j} V^{(i,k)}$ (but not invariant).*

(2) *$P_{kl}^{(i)}$ is zero on $V^{(j)}$, for $j \neq i$ and $\mathrm{Im}\ P_{kl}^{(i)}|_{V^{(i,l)}} \subset V^{(i,k)}$ and $P_{kl}^{(i)}: V^{(i,l)} \to V^{(i,k)}$ is an isomorphism of vector spaces.*

(3) *Take $x_1 \neq 0 \in V^{(i,1)}$ and let $x_k = P_{k1}^{(i)}(x_1) \in V^{(q,k)}$. Then $V(x_1) = \mathrm{span}(x_1, \dots, x_n)$is an invariant subspace where the representation V_1 is realised. Finally, if $x_{11}, x_{12}, \dots, x_{1m}$ is a basis of $V^{(i,1)}$, then*

$$V^{(i)} = V(x_{11}) \oplus V(x_{12}) \oplus \cdots \oplus V(x_{1m})$$

is a decomposition of $V^{(1)}$ into a direct sum of irreducibles.

Remark. The final statement says that finding irreducible subspaces is "as easy" as finding bases.

PROOF. First of all, $P_{kl}^{(i)}$ is defined for any V, and to compute them, we shall first consider an irreducible V, and then we use "linearity in V". Suppose V is irreducible, and let $f_1, \dots, f_r$ be a basis of V and suppose that $\rho_V(g) = (E_{pq}(g))$ with respect to $\{f_j\}$, then

$$\begin{aligned} P_{kl}^{(i)}(f_s) &= \frac{\dim V_i}{|G|} \sum_{g \in G} E_{lk}^{(i)}(g^{-1}) \rho_V(g)(f_s) \\ &= \frac{\dim V_i}{|G|} \sum_{g \in G} \sum_t E_{lk}^{(i)}(g^{-1}) E_{ts}(g) f_t \\ &= \begin{cases} 0 & \text{if } V \not\simeq V_i \\ f_k & \text{if } l = s \text{ and } V \simeq V_i \\ 0 & \text{if } l \neq s \text{ and } V \simeq V_i \end{cases} \end{aligned}$$

□

Lecture 12

18/10

First, we will finish that disastrously technical result we started last time. Recall we were considering the operators

$$P_{kl}^{(i)} = \frac{\dim V_1}{|G|} \sum_{g \in G} E_{lk}^{(i)}(g^{-1}) \rho(g)$$

Now, assume that V is irreducible and let $f_1, \dots, f_k$ be a basis then

$$P_{kl}^{(i)}(f_r) = \begin{cases} 0 & V \not\simeq V_i \\ f_k & V \simeq V_i \text{ and } l = r \\ 0 & V \simeq V_i \text{ and } l \neq r \end{cases}$$

Since for a general V, $\rho_V(g)$ are represented by block-diagonal matrices, $P_{kl}^{(i)}$ are also represented by block-diagonal matrices.

$$P_{kl}^{(i)} P_{st}^{(j)} = \begin{cases} 0 & i \neq j \\ P_{kt}^{(i)} & i = j, l = s \\ 0 & i = j, l \neq s \end{cases}$$

In particular $(P_{kk}^{(i)})^2 = P_{kk}^{(i)}$. We have the following

$$\rho_V(g) P_{kl}^{(i)} = \sum_m E_{mk}^{(i)}(g) P_{ml}^{(i)}$$

To show it, we show that it is true on basis elements from copies of V_j, $j \neq i$ we obtain $0 = 0$. If f_r is a basis element of a copy of V_i then

$$\begin{aligned} \rho_V(G) P_{kl}^{(i)}(f_r) &= \delta_{lr} \rho_V(g)(f_k) \\ &= \sum_m E_{mk}^{(i)}(g) P_{ml}^{(i)}(f_r) \\ &= \delta_{lr} \sum_m E_{mk}^{(i)}(g) f_m \end{aligned}$$

If we define $V^{(i,k)} = \operatorname{Im} P_{kk}^{(i)}$ then we have $V^{(i,k)} \subset V^{(i)}$ because we have shown that P_{kk} project onto the component part of V_i. Also we have $V^{(i)} = \oplus_k V^{(i,k)}$ - the sum is direct because we know that $P_{kk}^{(i)} P_{ll}^{(i)} = 0$ unless $k = l$ and if $v \in V^{(i,k)} \cap V^{(i,l)}$ then

$$v = P_{kk}^{(i)}(v_1) = P_{ll}^{(i)}(v_2)$$

and so

$$P_{kk}^{(i)}(v) = (P_{kk}^{(i)})^2(v_1) = P_{kk}^{(i)} P_{ll}^{(i)}(v_2)$$

so $v = P_{kk}^{(i)}(v_1) = (P_{kk}^{(i)})^2(v_1) = 0$.

We know that $P_{kl}^{(i)}$ acts like zero on $V^{(j)}$ for $j \neq i$ as prooved. We want to show that

$$P_{kl}^{(i)} : V^{(i,l)} \to V^{(i,k)}$$

is an isomorphism. Let $v \in \operatorname{Im} P_{ll}^{(i)} = V^{(i,l)}$ so $v = P_{ll}^{(i)}(v_1)$. We have

$$w = P_{kl}^{(i)}(v) = P_{kl}^{(i)} P_{ll}^{(i)}(v_1) = P_{kl}^{(i)}(v_1)$$

and

$$P_{kk}^{(i)}(w) = P_{kk}^{(i)} P_{kl}^{(i)}(v_2) = P_{kl}^{(i)}(v_1) = w$$

which implies that $w \in \operatorname{Im} P_{kk}^{(i)}$. Also the operator $P_{lk}^{(i)}$ is an inverse $V^{(i,k)} \to V^{(i,l)}$.

Now, let $x_1 \neq 0 \in V^{(i,1)}$ and define $x_k = P_{k1}^{(i)}(x_1) \in V^{(1,k)}$. Then $V(x_1) = \operatorname{span}\{x_1\}$ is an invariant subspace isomorphic to V_i. We have $x_1 = P_{11}^{(i)}(x_1)$. We have

$$\begin{aligned} \rho_V(g)(x_k) &= \rho_V P_{k1}^{(i)}(x_1) \\ &= \sum_m E_{mk}^{(i)}(g) P_{m1}^{(i)}(x_1) = \sum_m E_{mk}^{(i)}(g) x_m \end{aligned}$$

and so both statements hold. The final result also follows from this, but we wont show this because this proof is already too long.

Now let us move onto more useful and entertaining material.

12.1. Tensor products. Let V, W be vector spaces then we can form the tensor products $V \otimes W$. If $A: V \to V$ and $B: W \to W$ are linear operators then $\det(A \otimes B) = (\det A)^{\dim V} \cdot (\det B)^{\dim W}$. If (V, ρ_V) and (W, ρ_W) are representations of G, then $(V \otimes W, \rho_{V \otimes W})$ is a representation which has

$$\rho_{V \otimes W}(g) \coloneqq \rho_V(g) \otimes \rho_W(g)$$
$$\chi_{V \otimes W}(g) = \chi_V(g)\chi_W(g)$$

There is another construction: let (V, ρ) be a representation of G, then we have a representation (V^*, ρ_{V^*}) - where V^* is the space of linear functionals on V - given as

$$\rho_{V^*}(g)(x) \coloneqq x(\rho_V(g^{-1})(v)) \qquad x \in V^*$$

Check that this is a representation and that the character of this representation satisfies

$$\chi_{V^*}(g) = \overline{\chi_V}(g)$$

Lecture 13

25/10

13.1. Interlude: "averaging" over n-gons. Monday is a bank-holiday so the tutorial will be had at some other time. We will have the two classes that we will miss on Monday in the final week of term.

We will defer until some other time the material mentioned at the end of class last time. First, we will discuss the problem discussed in the first lecture about the regular n-gon and the averaging process that happens. It can be approached in a very elementary linear algebra method: we have an operator

$$T \begin{pmatrix} a_1 \\ a_2 \\ \dots \\ a_n \end{pmatrix} \to \begin{pmatrix} \frac{a_n + a_2}{2} \\ \frac{a_1 + a_3}{2} \\ \dots \\ \frac{a_{n-1} + a_1}{2} \end{pmatrix}$$

This operator can be written as $T = \frac{1}{2}(S + S^{-1})$ where S is a cyclic shift. If $\omega_k = \exp(2\pi i k/n)$ is an n-th root of unity, then

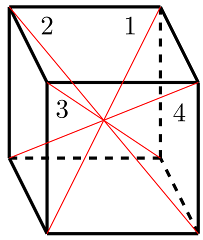
$$(1, \omega_k, \omega_k^2, \dots, 1)^T$$

is an eigenvector of S with eigenvalue ω_k. The same vector is an eigenvector of T with eigenvalue

$$\frac{1}{2}\left(\omega_k + \frac{1}{\omega_k}\right) = \cos\frac{2\pi k}{n}$$

If n is odd then only one of them survives, if n is even then there is "blinking" or oscillations.

If we consider the same problem except for a dice: the linear algebra approach is very difficult since we need the characteristic equation of a 6×6 matrix. We need to discuss the symmetries: it is a fact that the rotations of a cube are isomorphic to S_4: identify elements with the long diagonals of a cube. There are exactly 24 rotations



- The trivial rotation.
- 6 elements corresponding to rotations through 180° around the line connecting midpoints of opposite edges (correspond to transpositions).
- 6 elements corresponding to rotations through ±90° around the line connecting centers of opposite faces (4 cycles).
- 3 elements corresponding to rotations through 180° around the line connecting centers of opposite faces (pairs of diagonal transpositions).
- 8 elements for rotations through ±120° around the long diagonals (correspond to 3 cycles).

Having established this isomorphism, it naturally leads us to consider a 6-dimensional representation of S_4 in the space of functions on the sides of the cube. Let us compute the character of this representation

e	(12)	(12)(34)	(123)	(1234)
6	0^8	2	0	2

We will now figure out the multiplicities. Computing scalar products with irreducible characters, we see that this representation is

$$V \otimes \mathrm{sgn} \oplus U \oplus \mathrm{triv}$$

Now we view the sides as abstract elements of a set and consider functions on the sides. The trivial representation corresponds to constant functions where all numbers are equal. The $V \otimes \mathrm{sgn}$ corresponds to the case where numbers on opposite sides add up to zero. The remaining one corresponds to: even functions: either zero or integral numbers, where opposite sides are equal and all numbers add up to zero.

Now we want the eigenvalues. The triv case just gives us 1 since the averaging process does nothing. The case of $V \otimes \mathrm{sgn}$ gives zero because the first time we get something like $\frac{a+(-a)+b+(-b)}{4}$, so after the first iteration there is zero on every face. In the case of U we have by the conditions $2a + 2b + 2c = 0$ so $a + b = -c$, so take a face with a on it then after an iteration we have $\frac{-(a+b)-(a+b)+b+b}{4} = -\frac{a}{2}$ and similarly for the other faces so we have $-1/2$, whence

$$(\lambda = 0) \oplus (\lambda = -\frac{1}{2}) \oplus (\lambda = 1)$$

So if we apply our operator many times the first gives no contribution, the second one tends to zero and only the final one survives.

An exercise is to do the same thing except for the case where the values appear on the vertices of the cube, and averaging occurs with adjacent vertices.

14.1. Faithful representations.

Definition 8. Let G be a group and (V, ρ) a representation, then (V, ρ) is said to be **faithful** if $\ker \rho = \{e\}$.

Remark. Note that this is equivalent to: ρ is a faithful representation iff ρ is an embedding.

Theorem 11. *Let (V, ρ) be a faithful complex representation of a finite group G. Then all irreducible representations of G appear as constituents of $V^{\otimes n}$, $n \geq 1$.*

Before the proof, let us consider an example. Take $G = S_3$ and the representation is the irreducible two-dimensional representation U via symmetries of a triangle. Recall for S_3 we have

	e	(12)	(123)
triv	1	1	1
sgn	1	-1	1
U	2	0	-1
$U^{\otimes n}$	2^n	0	$(-1)^n$

We obtain

$$(\xi_{U^{\otimes n}}, \xi_{\text{triv}}) = \frac{1}{6}(2^n + 2(-1)^n)$$
$$(\xi_{U^{\otimes n}}, \xi_{\text{sgn}}) = \frac{1}{6}(2^n + 2(-1)^n)$$
$$(\xi_{U^{\otimes n}}, \xi_U) = \frac{1}{6}(2^{n+1} + 2(-1)^{n+1})$$

PROOF. Suppose that some complex irreducible representation W does not appear when we decompose $V^{\otimes n}$, $n \geq 1$ into irreducibles. First of all, let us choose for both V and W, the Hermitian scalar products invariant under the action of G. Let us pick both for V and for W an orthonormal basis with respect to the corresponding Hermitian product. I claim that if we work relative to these bases, then the matrix elements will have the property

$$E_{ij}(g^{-1}) = \overline{E_{ji}(g)}$$

This follows precisely from the fact that the product is invariant:

$$(\rho(g)e_k, e_l) = (e_k, \rho(g^{-1})e_l)$$

because we know that $(\rho(g)v, \rho(g)w) = (v, w)$. Now orthogonality relations for matrix elements look like

$$\frac{1}{|G|}\sum E_{ij}^{(1)}(g)\overline{E_{kl}^{(2)}(g)} = \begin{cases} 0 & \text{for non-isomorphic representations} \\ \frac{1}{\dim V}\delta_{il}\delta_{jk} & \text{when the two representations coincide} \end{cases}$$

the first case is relevant here because of the the hypothesis. The two E that appear in the formula above are the matrix elements with respect to the bases which are orthonormal for invariant Hermitian products. So under our assumption, the matrix elements of our representation of W are orthogonal to all matrix elements of all tensor powers of V.

Now, we would like to express the elements in terms of the tensor product. If $E_{ij}(g)$ are matrix elements of (V, ρ) then what are the matrix elements of

$(V^{\otimes 2}, \rho^{\otimes 2})$? Let $A = (a_{ij})$ relative to the basis $e_1, \dots, e_m$ and $B = (b_{ij})$ relative to $f_1, \dots, f_n$ then

$$\begin{aligned}(A \otimes B)(e_i \otimes f_j) = A(e_i) \otimes B(f_j) &= (\sum_k (a_{ki} e_k) \otimes (\sum_l b_{lj} f_l)) \\ &= \sum_{k,l} (a_{ki} b_{lj})(e_k \otimes f_l)\end{aligned}$$

So if $E_{ij}(g)$ are the matrix elements of (V, ρ) then $E_{ij}(g)E_{kl}(g)$ are the matrix elements of $(V^{\otimes 2}, \rho^{\otimes 2})$, $E_{ij}(g)E_{kl}(g)E_{pq}(g)$ are the matrix elements of $(V^{\otimes 3}, \rho^{\otimes 3})$ etc. By linearity, if

$$F(g) = P(E_{11}(g), E_{12}(g), \dots)$$

where P is a polynomial, then we have

$$\sum_{g \in G} F(g) \overline{E_{ij}^W(g)} = 0$$

where the E_{ij}^W are the matrix elements of W. Up until now we have not used the fact that the representation is faithful. Now, we work in $\mathbb{C}^{|G|}$ (with the standard Hermitian scalar product on $\mathbb{C}^{|G|}$) and identify F with its vector of values

$$F \longleftrightarrow (F(e), \dots, F(g), \dots)$$

and similarly

$$E_{ij}^W \longleftrightarrow (E_{ij}^W(e), \dots, E_{ij}^W(g), \dots)$$

The collection of matrix elements of V, viewed as functions on G, distinguishes elements from one another. As a consequence, every function on a group is a polynomial in those. This is analogous to Lagrange interpolation if we were to consider some finite subset on $\mathbb{R}^n$ or a very particular case of the Stone-Weierstraß theorem. Hence all matrix elements of W are polynomials in matrix elements of V, hence self-orthogonal which implies

$$E_{ij}^W(g) = 0$$

for all i, j which is a contradiction, a scalar product is positive-definite. □

Lecture 15

12/11 12pm

15.1. Set representations. Today the goal is to explain an important class of representations of finite groups which are usually called **set representations** or permutation representations.

Let G be a finite group and let M be a finite set, and let an (left) action of G be given on M. A group action gives rise to a representation of the set: if $\mathbb{F}$ is a field, we can construct a representation on a vector space denoted $\mathbb{F}M$ where the basis elements are indexed by elements of M i.e.

$$\mathbb{F}M = \{\sum c_m e_m \mid c_m \in \mathbb{F},\, m \in M\}$$

where $\mathbb{F}M$ has a basis e_m for $m \in M$. The action is given by $\rho(g)e_m = e_{g \cdot m}$, then we extend on all $\mathbb{F}M$ by linearity, and thus obtain a representation of G. Note that some of the representations that we have already discussed are examples of this sort of representation.

Examples:

(1) G a group, $M = \varnothing$ results in $\mathbb{F}M$ being the trivial representation.
(2) G a group, $M = G$, action by left multiplication $g \cdot h \coloneqq gh$ results in $\mathbb{F}M$ being the left regular representation.

(3) $G = S_n$ and $M = \{1, 2, \dots, n\}$ and the action is just the usual permutation action, then $\mathbb{F}M = \mathbb{F}^n$ is the representation by permuting the standard unit vectors in $\mathbb{F}^n$.

A set representation has the following features

(1) It contains a one-dimensional invariant subspace where the trivial representation is realised $(c \cdot \sum_{m \in M} e_m)$.
(2) The value of the character of $\mathbb{F}M$ on $g \in G$ is equal to the number of fixed points of g on M (all diagonal entries of $\rho(g)$ are zeroes and ones.)

Theorem 12. *Let M, N be two sets where G acts, then*

$$\mathrm{Hom}_G(\mathbb{F}M, \mathbb{F}N) \simeq \mathbb{F} \cdot (M \times N)/G \quad \textit{as vector spaces}$$

PROOF. Take $\alpha \in \mathrm{Hom}(\mathbb{F}M, \mathbb{F}N)$. Let α be represented by a matrix (a_{mn}) with

$$\alpha(e_m) = \sum_{n \in N} a_{nm} e_n$$

Now we want to look at the intertwining operators: recall

$$\alpha \text{ is an intertwining operator } \Leftrightarrow (\rho_N(g)\alpha = \alpha\rho_M(g))$$

so we have

$$\rho_N(g)\alpha(e_m) = \alpha\rho_M(g)(e_m)$$

Then the left-hand side is

$$\rho_N(g) \sum_n a_{nm} e_n = \sum_n a_{nm} e_{g \cdot n}$$

and the right-hand side

$$\alpha(e_{g \cdot m}) = \sum_n a_{n\, g \cdot m} e_n$$

so the condition for it to be an intertwining operator is

$$\begin{aligned} \sum_n a_{nm} e_{g \cdot n} &= \sum_n a_{n\, g \cdot m} e_n \\ &= \sum_k a_{g \cdot k\, g \cdot m} e_{g \cdot k} \end{aligned}$$

so we have $a_{nm} = a_{g \cdot n\, g \cdot m}$ for all m, n, g, i.e. it is constant on each orbit. □

So what are some consequences of this result

(1) Let M be any set with a G-action and $N = \{*\}$ then we have

$$\dim \mathrm{Hom}_G(\mathbb{F}M, \mathbb{F}N) = |M/G|$$

which has the following informal explanation: $\mathbb{F}M$ has a copy of the trivial representation for each orbit of G.

(2) Let $G = S_n$ and $M = \{1, 2, \dots, n\}$ and $N = \mathcal{P}_2 M$ (all two element subsets of M). We have $|M| = n$ and $|N| = \binom{n}{2}$, and

$$\dim \mathrm{Hom}_{S_n}(\mathbb{F}M, \mathbb{F}M) = |(M \times M)/S_n| = 2$$

because the orbits on pairs are $\{(k, k)\}$ and $\{(p, q),\, p \neq q\}$. In the case

$$\dim \mathrm{Hom}_{S_n}(\mathbb{F}M, \mathbb{F}N) = |(M \times N)/S_n| = 2$$

because elements are $M \times N = \{(k, \{p, q\})\}$ and the orbits are $\{(k, \{p, q\}), k \in \{p, q\}\}$ and $\{(k, \{p, q\}), k \notin \{p, q\}\}$ and finally

$$\dim \mathrm{Hom}_{S_n}(\mathbb{F}N, \mathbb{F}N) = 3 \qquad (n \geq 4)$$

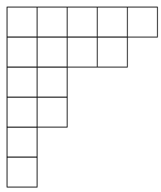
because the 2-element subsets can coincide, can overlap by one or be disjoint. We have

$$\mathbb{C}M = \text{triv} \oplus V$$
$$\mathbb{C}N = \text{triv} \oplus V \oplus W$$

Lecture 16

12/11 4pm

16.1. Representations of S_n via set representations. Conjugacy classes of S_n are in correspondence with partitions of n, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ where $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. These partitions are usually represented by **Young diagrams**



where in the above diagram the row k (from the top)[9] contains λ_k unit squares. λ, a partition of n, corresponds to a set M_λ on which S_n acts. M_λ is the set of Young diagrams with the underlying diagram λ. A **Young tableau** is a numbering of unit squares of λ by $1, \ldots, n$ up to row permutations.

An example: take $n = 3$ then we have things like $\boxed{1}\boxed{2}\boxed{3}$, $\boxed{1}\boxed{3}\boxed{2}$, $\boxed{2}\boxed{3}\boxed{1}$, ... but these are all the same by our conventions, so there is just one distinct Young tableau of this type. We have three distinct tableaux of the following type $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$ and $\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array}$ and there are six distinct tableaux of the following type:

$$\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 3\\ \hline\end{array}\,,\ \begin{array}{|c|}\hline 1\\ \hline 3\\ \hline 2\\ \hline\end{array}\,,\ \begin{array}{|c|}\hline 2\\ \hline 1\\ \hline 3\\ \hline\end{array}\,,\ \begin{array}{|c|}\hline 2\\ \hline 3\\ \hline 1\\ \hline\end{array}\,,\ \begin{array}{|c|}\hline 3\\ \hline 1\\ \hline 2\\ \hline\end{array}\,,\ \begin{array}{|c|}\hline 3\\ \hline 2\\ \hline 1\\ \hline\end{array}$$

The first type just corresponds to the trivial representation. The second group of three Young tableaux corresponds to $\mathbb{C}^3 \simeq \text{triv} \oplus U$ and the final one corresponds to the left-regular representation which is $\text{triv} \oplus U \oplus U \oplus \text{sgn}$.

Lemma 13. *Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k), \mu = (\mu_1 \geq \cdots \geq \mu_l)$ both be partitions of n then*

$$\chi_{\mathbb{C}M_\lambda}(\mu) = \textit{ coefficient of } x_1^{\lambda_1} \ldots x_k^{\lambda_k} \textit{ in } \prod_{i=1}^{l} (x_1^{\mu_i} + x_2^{\mu_i} + \cdots + x_k^{\mu_i})$$

PROOF. The number of fixed products of a representation of type μ is equal to the number of ways to assign to cycles in μ the rows within which those cycles permute elements. These ways naturally correspond to choices of $x_i^{\mu_j}$ in each bracket that altogether assemble into our monomial. $\square$

Here are some characters for various set representations of S_4.

[9]This is the English notation; the French notation is this inverted i.e. go bottom-up rather than top-down.

	e	(12)	$(12)(34)$	(123)	(1234)	Decomposition
M_4	1	1	1	1	1	triv
$M_{3,1}$	4	2	0	1	0	triv $\oplus V$
$M_{2,2}$	6	2	2	0	0	triv $\oplus U \oplus V$
$M_{2,1,1}$	12	2	0	0	0	triv $\oplus U \oplus V \oplus V \oplus V'$
$M_{1,1,1,1}$	24	0	0	0	0	all irrep.'s with multiplicity

Heuristically: the idea that can be seen from the table above is that when we move from each set representation down to the next, then if we "chop off" the previous representation (including multiplicities) then we get a new representation.

An example of the calculation of the result from the theorem: take $\lambda = \mu = (2,2)$ then we have

$$(x_1^2 + x_2^2)(x_1^2 + x_2^2) = x_1^4 + \underbrace{2x_1^2x_2^2}_{\lambda} + x_2^4$$

Now, let us list some features of this construction

(1) For each partition λ, there is an irreducible V_λ which appears in $\mathbb{C}M_\lambda$ with multiplicity 1 and does not appear in "smaller" $\mathbb{C}M_{\lambda'}$. What do we mean by smaller? We say that $\lambda_1 \geq \cdots \geq \lambda_k$ is bigger than $\lambda'_1 \geq \cdots \geq \lambda'_l$ if

$$\lambda_1 \leq \lambda'_1$$
$$\lambda_1 + \lambda_2 \leq \lambda'_1 + \lambda_2$$
$$\lambda_1 + \lambda_2 + \lambda_3 \leq \lambda'_1 + \lambda'_2 + \lambda'_3$$
$$\vdots$$

This is the most important property of set representations and it is how one studies representations of the symmetric group in general.

Next class will be a tutorial.

Lecture 17

19/11 12pm

17.1. Construction of the irreducible representations of S_n over a field of characteristic zero. There are also **Young tabloids**, which are Young tableaux up to row permutations e.g. $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}$ as Young tabloids but not Young tableaux. For λ a Young diagram then we denote T_λ the set of all Young tableaux of shape λ and by M_λ the set of all Young tabloids of shape λ. Both T_λ and M_λ have an action of S_n on them.

$\mathbb{C}T_\lambda$ is isomorphic to the left regular representation[10]. There is a natural map $\mathbb{C}T_\lambda \to \mathbb{C}M_\lambda$ that sends a tableau to its equivalence class of tabloids. For each Young tableau t of shape λ, let us define an element $v_t \subset \mathbb{C}T_\lambda$ as follows

$$v_t = \sum_{\sigma \in c_t} \operatorname{sign}(\sigma) \cdot e_{\sigma(t)}$$

where c_t is the set of elements of S_n permuting elements inside the columns in the Young tableau t, for instance say that your Young tableau was a horizontal strip then the subgroup would be trivial.

Here are some facts, some of which are easy to show, others require some hard work

(1) $\underset{\substack{t\text{: tableau} \\ \text{of shape } \lambda}}{\operatorname{span}}(v_t) \subset \mathbb{C}T_\lambda$ is an invariant subspace for the S_n-action.

(2) The image of $\operatorname{span}(v_t)$ in $\mathbb{C}M_\lambda$ is an irreducible representation V_λ of S_n.

[10]As far as I can see, this is only true if we take $\lambda = (n-1,1)\ldots$

(3) V_λ are pairwise non-isomorphic for different λ, and each irreducible representation of S_n is isomorphic to one of them.

These describe in quite a nice constructive way the irreducible representations. The proofs are in Fulton & Harris's textbook[11].

Now, recall the stuff from the last homework about A_5

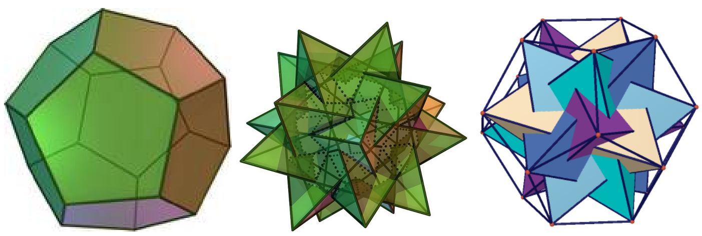
- the conjugacy classes in S_5 that are not odd are $e, 1^5, 2^2 1, 31^2, 5$. In A_5, the 5-cycles split, we will denote them by 5_1 (corresponds to the one containing (12345)) and 5_2 (containing (21345)). One way to obtain irreducible representations is to try to restrict the representations of S_5 to A_5 and see what happens. Recall from the tutorial the character table that we had for S_5 and restrict to A_5

	1^5	2^21	31^2	5_1	5_2
#	1	15	20	12	12
V_5	1	1	1	1	1
$V_{4,1}$	4	0	1	-1	-1
$V_{3,2}$	5	1	-1	0	0
$V_{3,1,1}$	6	-2	0	1	1

By computing characters with themselves we can see that all of these are still irreducible except for the last one where the product with itself is 2. $V_{3,1,1}$ is more difficult to understand than the others, we will discuss them in the second lecture today. In the last few minutes we will discuss a fact that we will need

Lemma 14. *A_5 is isomorphic to the group of all rotations of a dodecahedron.*

Remark. Recall when we showed something similar for the cube we tried to find four things where one of them was fixed under every action of the group. It is much harder to see it in the case of the dodecahedron, but one can embed a compound of five tetrahedra in it to group the points of faces into five groups. The images below show the dodecahedron, the chiroicosahedron (compound of five tetrahedra) and finally the chiroicosahedron embedded in the dodecahedron.



Lecture 18

19/11 4pm

18.1. Decomposing A_5. Continuing about A_5. We figured out that all except $V_{3,1,1}$ are irreducible. We took the scalar product of this and got 2, so we expect to obtain a direct sum of two irreducible representations.

Claim. $V_{3,1,1}$ as a representation of A_5 is isomorphic to the sum of two three dimensional irreducibles.

[11] Lecture 3.

PROOF. Since A_5 is a subgroup of S_5 we can use some "external" information to learn about the irreducibles here - act on it by transpositions from S_5. If U is an invariant subspace, where one of the irreducibles is realised, of $V_{3,1,1}$, consider $\rho((12))U \subset V_{3,1,1}$. This subspace is also A_5-invariant: if $g \in A_5$,

$$\begin{aligned}\rho(g)\rho((12))U = \rho((12))\rho((12))^{-1}\rho(g)\rho((12))U &= \rho((12))\rho(\underbrace{(12)^{-1}g(12)}_{\in A_5})U \\ &= \rho((12))U\end{aligned}$$

$U \cup \rho((12))U$ is an invariant subspace of U, so since U is irreducible it is either zero or all of U. It cannot be the whole space because then we would have to have $\rho((12))U = U$ which would mean that U is invariant under the action of transpositions, but then U would be invariant under the action of all of S_5, i.e. it is an S_5-invariant which is a contradiction. Therefore we have proved that (because then U and $\rho((12))U$ are disjoint, so the sum is direct)

$$V_{3,1,1} \simeq U \oplus \rho((12))U$$

(there is nothing else because $(\chi, \chi) = 2$) $\square$

In fact we have prooved something more general here: if the restriction from G to a subgroup H of index two of an irreducible representation is not irreducible, then it is isomorphic to the sum of two irreducibles of the same dimension (we used transpositions in the proof, but actually it does not matter).

Another consequence is

$$\chi_{U_1}(g) = \chi_{U_2}((12)^{-1}g(12)) \tag{18.5}$$

because we can write $\rho(g)\rho((12))U = \rho((12))\rho((12)^{-1}g(12))U$. This is sort of like moving from one vector space to another vector space, akin to moving from A_5 up to S_5. One should take some time to contemplate the above equation to try to understand this subtlety here.

Now we try to fill in the rest of the table started last time. For all conjugacy classes of S_5 that do not split in A_5, this formula (18.5) means that $\chi_{U_1}(g) = \chi_{U_2}(g)$.

	1^5	2^21	31^2	5_1	5_2
#	1	15	20	12	12
$V_{3,1,1}$	6	-2	0	1	1
U_1	3	-1	0	a	$1-a$
U_2	3	-1	0	$1-a$	a

where the first instance of -1 are chosen because $-1 - 1 = -2$ which is the character in the entry for $V_{3,1,1}$ and so on. But we cannot do this with the 5-cycles. So we use the fact that $\chi_{U_2}(5_1) = \chi_{U_2}(5_2)$ since $(12)^{-1}5_1(12) = 5_2$. To determine the value of a we use the fact that the inner product must be one, we obtain by calculation

$$24a^2 - 24a = 24 \Rightarrow a^2 - a + 1 \Rightarrow a = \frac{1 \pm \sqrt{5}}{2}$$

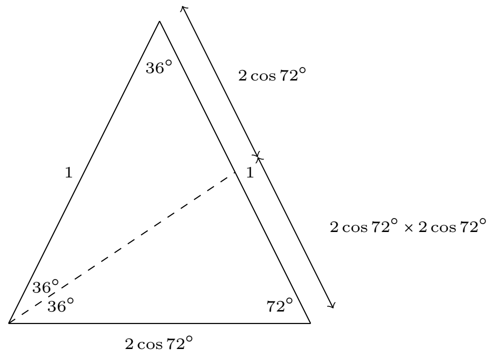
and they ought to be different - we need to be able to label the two irreducibles after all, if they are distinct. So the final table is

	1^5	$2^2 1$	31^2	5_1	5_2
#	1	15	20	12	12
$V_{3,1,1}$	6	-2	0	1	1
U_1	3	-1	0	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
U_2	3	-1	0	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$

Now we would like to be able to look at what these irreducibles U_1, U_2 actually are, we will use the dodecahedron to do this. The conjugacy classes in the group of rotations of the dodecahedron are

- e
- rotations about the axis through opposite vertices, $\pm 120°$ which correspond to three cycles.
- rotations about the axis the line connecting midpoints of opposite edges through $180°$ which corresponds to the product of disjoint two-cycles.
- rotations about the edges connecting the centres of opposite faces, through the angles $\pm 72°$ and through $\pm 144°$, where 5_1 corresponds to $\pm 72°$ and the other to 5_2.

For the rotation through $72°$, the character of the three-dimensional representation is $1 + 2\cos 72°$. To finish this lecture lets do some elementary geometry that you might have seen in high-school



This small geometric argument tells us that

$$2\cos 72° + 4(\cos 72°)^2 = 1$$

which tells us that $2\cos 72° = \frac{-1+\sqrt{5}}{2}$ so $1 + 2\cos 72° = \frac{1+\sqrt{5}}{2}$ so we see this corresponds with the table entries from earlier.

Lecture 19

22/11

19.1. Dedekind-Frobenius determinants. The goal today is to see some applications of representations to group theory. This will also cover group algebras which I mentioned before. To make the transition we will first discuss again that problem from the first lecture about the Dedekind-Frobenius determinants.

Let G be a group, and introduce formal indeterminates x_g indexed by elements $g \in G$. First label the rows and columns of a matrix by elements $g \in G$ so that the entry in the matrix at position (g, h) is x_{gh}. Next, we permute the columns of the matrix so that the entry at that position is now $x_{gh^{-1}}$. Let us take the underlying space $\mathbb{C}G$ of the left regular representation. Then this matrix is[12]

$$\sum_{g\in G} x_g \rho(g)$$

(relative to the basis $\{e_g\}$ of $\mathbb{C}G$). Consider

$$\sum_{g\in G} x_g \rho(g)(e_h) = \sum_{g\in G} x_g e_{gh} = \sum_{g\in G} x_{(gh)h^{-1}} e_{gh} = \sum_{g'\in G} x_{g'h^{-1}} e_{g'}$$

so we precisely recover the columns of our matrix - "put the coordinates of the images of the basis vectors in the columns of the matrix". So our matrix has this nice meaning in terms of the left-regular representation. When we decompose $\mathbb{C}G$ into irreducibles, each one appears with multiplicity equal to its dimension. Let us choose a basis of $\mathbb{C}G$ corresponding to such a decomposition. Each $g \in G$ will be represented by block matrices, each block being the matrix representing g in the corresponding irreducible representation. Therefore $\sum_{g\in G} x_g \rho(g)$ is also represented by a block diagonal matrix, and the determinant is equal to the product of determinants of blocks, that is

$$\prod_{i=1}^{s} \Big[\det(\sum_{g\in G} x_g \rho_i(g)) \Big]^{\dim V_i}$$

where $(V_i, \rho_i)_{,i=1,\ldots,s}$ are the complex irreducible representations of G. In fact one can reconstruct the Cayley table of a group if you have the above quantity.

Proposition 15. *This factorisation is a factorisation into irreducibles.*

We want to proove that $\det \sum_{g\in G} x_g \rho_i(g)$ is irreducible for each $i = 1, \ldots s$. Let $n_i = \dim V_i$, we shall proove that there exists

$$x_g = x_g(a_{11}, a_{12}, \ldots, a_{an}, \ldots, \ldots, a_{nn})$$

such that $\sum_{g\in G} x_g \rho_i(g) = (a_{jk})_{j,k=1,\ldots,n}$. It would reduce prooving irreducibility to proving irreducibility of

$$\det \begin{pmatrix} a_{11} & \ldots & a_{1n_i} \\ \vdots & & \vdots \\ a_{n_i 1} & \ldots & a_{n_i n_i} \end{pmatrix}$$

If $\det(a_{pq}) = F_1 F_2$ then F_1 has degree 1 in a_{11} and F_2 has degree 0 in a_{11} and if F_2 is not a constant polynomial then it is easy to see that it must vanish at some point. (I didn't understand the rest of this argument ☹)

Lecture 20

26/11 12pm

20.1. The group algebra. We continue where we left off last time. The remaining goal was to proove that there exists coefficients $\alpha_g = \alpha_g(a_{ij})$ such that $\sum_{g\in G} \alpha_g \rho_i(g) = (a_{ij})$. We will discuss this in the context of a group algebra. Consider $\mathbb{F}G$ and let us work over an arbitrary field $\mathbb{F}$ and take the underlying space of the left regular representation. This space has a basis indexed by group elements $\{e_g\}$. Let us introduce a product on this space $\mu: \mathbb{F}G \otimes \mathbb{F}G \to \mathbb{F}G$ as follows

$$e_g \otimes e_h \mapsto e_{gh}$$

and once we know how to find products of basis elements, then we can find all elements by linearity of course. So, of course this product is associative, so this vector space $\mathbb{F}G$ becomes an associative algebra over the field $\mathbb{F}$. Once you have

[12] Because of the way the left regular representation is defined $\rho(g)e_h = e_{gh}$, the column of the matrix of $\rho(g)$ in the basis $\{e_i\}$ corresponding to h has zero everywhere except at the row corresponding to gh, where it has a one.

the notion of an associative algebra, a notion which is important is that of a module over an associative algebra. Consider for instance left-modules over $\mathbb{F}G$[13]. In some sense if you know how to classify left-modules over an associative algebra then you can answer any question in linear algebra. In particular, there is a one-to-one correspondence between left $\mathbb{F}G$-modules and representations (V,ρ) of G over the field $\mathbb{F}$. In particular, if M is our left-module and V is the vector space, with matrices $\rho(g)$ for $g \in G$ then

$$\alpha(\sum c_g e_g \otimes m) \longrightarrow V \coloneqq M,\ \rho(g)(m) = \alpha(e_g \otimes m)$$

Now, let $\mathbb{C} = \mathbb{F}$ and let $V_1, \dots, V_s$ be irreducible representations of G. Then we have an algebra homomorphism

$$\mathbb{C}G \to \mathrm{Mat}_{n_1}(\mathbb{C}) \oplus \cdots \oplus \mathrm{Mat}_{n_s}(\mathbb{C})$$

and this map takes

$$e_g \mapsto (\rho_1(g), \rho_2(g), \dots, \rho_s(g)) \tag{20.6}$$

my claim that I want to proove is

Claim. The map (20.6) is an isomorphism.

PROOF. As vector spaces, these have the same dimension $|G| = n_1^2 + \cdots + n_s^2$, so it is enough to proove that this map is a surjection. Suppose that the image is a proper subspace of

$$\mathrm{Mat}_{n_1}(\mathbb{C}) \oplus \cdots \oplus \mathrm{Mat}_{n_s}(\mathbb{C})$$

This means that there is a linear function on the above space that is identically zero on the image of the map. Each linear function is

$$\sum_{p=1}^{s} \sum_{i,j=1}^{n_p} \alpha_{ij}^{(p)} a_{ij}^{(p)}$$

where $((a_{ij}^{(1)}), \dots, (a_{ij}^{(s)}))$ is the element on which we compute the linear function. There exists $\alpha_{ij}^{(p)}$ not all equal to zero such that

$$\sum_{\substack{p=1,\dots,s \\ i,j=1,\dots,n_p}} \alpha_{ij}^{(p)} E_{ij}^{(p)}(g) = 0$$

for all $g \in G$.[14] Multiplying by $E_{kl}^{(r)}(g^{-1})$ and summing over all $g \in G$ we obtain

$$\sum \alpha_{ij}^{(p)} \sum_{g \in G} E_{ij}^{(p)}(g) E_{kl}^{(r)}(g^{-1}) = 0$$

Now recall our old identities, this vanishes almost everywhere. In particular, this term $\sum_{g\in G} E_{ij}^{(p)}(g) E_{kl}^{(r)}(g^{-1})$ is equal to zero except for the case $p = r$, $i = l$, $j = k$, so

$$\alpha_{lk}^{(r)} \frac{1}{n_r} = 0$$

[13] Recall, if A is an associative algebra over $\mathbb{F}$, then M is said to be a left A-module if

- M is a vector space over $\mathbb{F}$.
- There is a map $\alpha: A \otimes M \to M$ i.e. there is an "action of A on M" - the following diagram is commutative

$$\begin{tikzcd}
A \otimes A \otimes M \arrow[r, "\mathrm{id}_A \otimes \alpha"] \arrow[d, "\mu \otimes \mathrm{id}_M"'] & A \otimes M \arrow[d, "\alpha"] \\
A \otimes M \arrow[r, "\alpha"] & M
\end{tikzcd}$$

For example, $\mathrm{Mat}_n(\mathbb{F})$ is an associative algebra and $\mathbb{F}^n$ is a module over $\mathrm{Mat}_n(\mathbb{F})$.

[14] Because it is zero on the image.

so $\alpha_{lk}^{(r)} = 0$. □

So the map

$$\mathbb{C}G \to \mathrm{Mat}_{n_1}(\mathbb{C}) \oplus \cdots \oplus \mathrm{Mat}_{n_s}(\mathbb{C})$$

is an isomorphism, and thus if we can consider an individual component map $\mathbb{C}G \to \mathrm{Mat}_{n_i}(\mathbb{C})$ with $e_g \mapsto \rho_i(g)$ surjective, and this is precisely what we need: this means that the matrix is a combination of $\rho_i(g), g \in G$. This completes the proof of the result that we started on Thursday. Also, it shows some of the uses that group algebras have.

Lecture 21

26/11 4pm

21.1. Centre of the group algebra. Let us continue where we left off. A notion that arises when discussing associative algebras, is the **centre**

Definition 9. If A is an associative algebra (over a field $\mathbb{F}$), then its centre $Z(A)$ is the sub-algebra

$$\{z : za = az, \text{ for all } a \in A\}$$

Let me start with an example, which is actually very important. Take $A = \mathrm{Mat}_n(\mathbb{F})$, then[15]

$$Z(A) = \{\lambda \cdot \mathrm{Id},\ \lambda \in \mathbb{F}\}$$

Take the group algebra of a finite group $\mathbb{C}G$, which we already showed is isomorphic to

$$\mathrm{Mat}_{n_1}(\mathbb{C}) \oplus \cdots \oplus \mathrm{Mat}_{n_s}(\mathbb{C})$$

We have

$$Z(\mathbb{C}G) \simeq \underbrace{Z(\mathrm{Mat}_{n_1}(\mathbb{C})) \oplus \cdots \oplus Z(\mathrm{Mat}_{n_s}(\mathbb{C}))}_{\mathbb{C}^s}$$

We want to compute it explicitly. We just need to look at the basis elements clearly

$$\Big(\sum_{g\in G} c_g e_g\Big)e_h = e_h\Big(\sum_{g\in G} c_g e_g\Big) \quad \text{for all } h \in G$$

then we have $\sum_{g\in G} c_g e_{gh} = \sum_{g\in G} c_g e_{hg} = \sum_{g'\in G} c_{h^{-1}g'h} e_{g'h}$ so we see that in order for our elements to be central, we should have $c_g = c_{h^{-1}gh}$ for all h. This computation shows that the constants are the same on the conjugacy classes. Thus the dimension of the centraliser is equal to the number of conjugacy classes, and we have already shown that $Z(\mathbb{C}G)$ is isomorphic to the number of distinct irreducible representations' copies of $\mathbb{C}$. More precisely, for each conjugacy class $\mathbf{C}$ of G, the element $E_{\mathbf{C}} = \sum_{g\in\mathbf{C}} e_g$ is in $Z(\mathbb{C}G)$ and these elements form a basis of the centre of the group algebra. So, now we want to discuss some consequences of this result

- For each conjugacy class $\mathbf{C}$ of G, $E_{\mathbf{C}}$ acts by a scalar (by Schur's lemma) in every irreducible representation of G. For each irreducible representation, this scalar $\lambda_{i,\mathbf{C}}$ is an algebraic integer (we shall proove this second fact later).

[15] Denote $E_{i,j} = \{(e_{jk}) : e_{ij} = 1, 0 \text{ otherwise}\}$ then for $C \in Z(A)$, $CE_{i,i}$ is the matrix with the ith column of C and zero elsewhere and $E_{i,i}C$ is the matrix with the ith row of C and zero elsewhere, so C is diagonal. Also $C(E_{i,j} + E_{j,i})$ is the matrix with columns i and j swapped and zero elsewhere, and compare with the same except by left multiplication of C, and the result follows.

- We have

$$\begin{aligned}\operatorname{tr}(\rho_i(E_{\mathbf{C}})) &= \sum_{g\in\mathbf{C}} \operatorname{tr}\rho_i(g) = |\mathbf{C}|\cdot\chi_i(\mathbf{C})\\ \operatorname{tr}(\rho_i(E_{\mathbf{C}})) &= n_i\lambda_{i,\mathbf{C}}\\ \Rightarrow \lambda_{i,\mathbf{C}} &= \frac{|\mathbf{C}|\cdot\chi_i(\mathbf{C})}{n_i}\end{aligned}$$

Let me use this result to proove the following

Lemma 16. *Suppose that* $\gcd(|\mathbf{C}|, n_i) = 1$, *then either* $\chi_i(\mathbf{C}) = 0$ *or* $\rho_i(g)$ *for* $g \in \mathbf{C}$ *are all scalar multiples of the identity.*

PROOF. If $\gcd(|\mathbf{C}|, n_i) = 1$ then there are integers a, b so that $a\#\mathbf{C} + bn = 1$. If $\lambda_{i,\mathbf{C}}$ is an algebraic integer so $a\cdot\lambda_{i,\mathbf{C}}$ is an algebraic integer so

$$\frac{(1-bn_i)\cdot\chi_i(\mathbf{C})}{n_i}$$

is an algebraic integer so

$$\frac{\chi_i(\mathbf{C})}{n_i} - \chi_i(\mathbf{C})$$

is an algebraic integer so $\chi_i(\mathbf{C})/n_i$ is an algebraic integer and

$$\left|\frac{\chi_i(\mathbf{C})}{n_i}\right| \le 1$$

by the triangle inequality, since the $\chi_i(\mathbf{C})$ are n roots of unity. If $|\frac{\chi_i(\mathbf{C})}{n_i}| = 1$ then it's a scalar matrix (prooved in homework). If $\frac{\chi_i(\mathbf{C})}{n_i} < 1$ then consider the polynomial of smallest degree with integer coefficients and root $\frac{\chi_i(g)}{n_i}$. Then all other roots are also averages of roots of unity, so of absolute value ≤ 1. The constant term on the polynomial is the product of all the roots, and is an integer, also it has absolute value strictly less that one. The only possibility is that it is zero, as required. □

In the remaining minute and a half, I just want to remind you about two statements from group theory, which will be needed on Thursday. The first statement is that

> A p-group has a non-trivial centre.

To recall why this is true, remember that g is in the centre $\Leftrightarrow$ the conjugacy class of g consists of g only. The second statement is

> Sylow's theorem: if the number of elements in G is $p^m k$ and p is prime, and k, p are coprime, then there is a subgroup of G of order p^m.

Lecture 22

29/11

Suppose that $\mathbf{C}$ is a conjugacy class of G, and $g \in \mathbf{C}$ then we know from last time that

$$\frac{\#\mathbf{C}\cdot\chi_{\rho_i}(g)}{n_i}$$

is an algebraic integer, where (V_i, ρ_i) is irreducible, with $\dim V_i = n_i$. If $\gcd(\#\mathbf{C}, n_i) = 1$ then $\chi_{\rho_i}(g) = 0$ or $\rho_i(g)$ is a scalar matrix. We can use this result to proove the following theorem

Theorem 17. *Suppose that for a conjugacy class* $\mathbf{C} \neq \{e\}$ *of* G*, we have* $\#\mathbf{C} = p^k$*, where* p *is a prime, then* G *is not simple.*

PROOF. Suppose that G has a non-trivial normal subgroup and let $g \neq e$ be in $\mathbf{C}$. Then if we evaluate the character of the left-regular representation[16]

$$0 = \chi_{\mathrm{reg}}(g) = \sum_{i=1}^{s} n_i \chi_i(g) = 1 + \sum_{i=2}^{s} n_i \chi_i(g)$$

(where the 1 in the sum corresponds to the copy of the trivial representation that you get). If n_i is not divisible by p, then $\gcd(\#\mathbf{C}, n_i) = 1$, and therefore $\chi_i(g) = 0$ or $\rho_i(g)$ is a scalar matrix. If G were simple then it cannot be a scalar matrix because of the homework question: we know that if some elements of G are represented by a scalar matrix on V_i, then all these elements form a normal subgroup in G. So we must have $\chi_i(g) = 0$. Therefore

$$0 = 1 + \sum_{i=2}^{s} n_i \chi_i(g) = 1 + \sum_{\substack{n_i \text{ such that} \\ p | n_i}} n_i \chi_i(g)$$

which is equivalent to

$$\sum_{\substack{n_i \text{ such that} \\ p | n_i}} \frac{n_i}{p} \chi_i(g) = -\frac{1}{p}$$

which is impossible because the left-hand side is an algebraic integer (with leading coefficient 1) and the right-hand side is a rational number that is not an integer. □

Remark. I will try to motivate the idea of a simple group. It is a natural question to try to classify all finite Abelian groups up-to isomorphism. This is too hard, so we instead focus on the simple groups and then see how all the other finite Abelian groups can be constructed out of this - the simple groups form the "building blocks" - recall that given a normal subgroup we can quotient it out and get a smaller group, this is the point &c.

22.1. Burnside's $p^a q^b$-theorem. Let us proove a simple consequence of the previous theorem.

Theorem 18 (Burnside's $p^a q^b$-theorem)**.** *Let* p, q *be prime numbers. Then a group of order* $p^a q^b$ *is not simple.*

Remark. This proof relies on two results that we mentioned at the end of the last class: Sylow's theorem, and existence of a non-trivial centraliser of a p-group (proof: Rose, THM 4.28)

PROOF. Let G be a group of order $p^a q^b$. Via Sylow's theorem: G has a subgroup H of order q^b, and via the other fact: H has a non-trivial centre. Let h be in the centre of H, then Z_h in G contains H. So, for C_h in G we have

$$\#C_h = \frac{\#G}{\#Z_h} = p^k$$

because $p^a q^b = \#G$ and Z_h contains H so its order is divisible by q^b. So the result follows from the previous theorem. □

Burnside's theorem says that a group of order $p^a q^b$ is solvable. Recall that a group G is said to be solvable if the following condition is satisfied: there exists a chain of subgroups

$$G = G_0 \supset G_1 \supset G_2 \supset \cdots \supset G_k = \{e\}$$

[16]Recall the action of the left-regular representation - the trace is zero unless $g = e$.

such that G_{i+1} is normal in G_i for each i and G_i/G_{i+1} is Abelian. It is a fact that if G is a group, and N is a normal subgroup and N and G/N are solvable, then G solvable. This can be used to proove that a group of order $p^a q^b$ is solvable by induction.

Lecture 23

3/12 12pm

Monday next week, there will be only one class - at 12pm. One thing that remains unprooved, that I used a number of times is the following

Theorem 19. *Let G be a finite group, and let (V, ρ) be an irreducible representation over $\mathbb{C}$. Then $\dim V$ divides $\#G$.*

PROOF. For each $g \in G$, we have

$$\frac{\#C_g \cdot \chi(g)}{\dim V} \quad \text{is an algebraic integer}$$

Consider the following sum

$$\sum_{\substack{g \text{ representatives} \\ \text{of conj. classes}}} \frac{\#C_g \chi(g)}{\dim V} \overline{\chi(g)} = \sum_{g \in G} \frac{\chi(g)\overline{\chi}(g)}{\dim V} = \frac{\#G}{\dim V}(\chi, \chi) = \frac{\#G}{\dim V}$$

and this is an algebraic integer, therefore an integer, therefore $\dim V$ divides $\#G$. □

This already puts some severe restrictions on the dimensions of irreducible representations. There is a question in the sample paper that deals with this question and the converse. That question is just a toy example and not something hugely important though.

One thing that I stated last week without proof is that

$$\frac{\#C_g \cdot \chi(g)}{\dim V} \quad \text{is an algebraic integer}$$

The proof of this fact is not from representation theory, although the expression itself is something from representation theory. The idea is the following: let $\mathbb{C}G$ be a group algebra, the centre of $\mathbb{C}G$ is spanned by the elements

$$E_{\mathbf{C}} = \sum_{g \in \mathbf{C}} e_g$$

Moreover, if we just consider $\oplus \mathbb{Z} E_{\mathbf{C}}$, which is a commutative sub-ring of the group algebra, because consider $E_{\mathbf{C}_1} \cdot E_{\mathbf{C}_2}$ is in the centre of $\mathbb{C}G$, hence a $\mathbb{C}$-linear combination of $E_{\mathbf{C}}$, but since $E_{\mathbf{C}_1} E_{\mathbf{C}_2} \subseteq \oplus_{g \in G} \mathbb{Z} e_g$, the coefficients are integer. We need the following

Lemma 20. *If R is a commutative ring which is finitely generated as an Abelian group, then each $x \in R$ is an "algebraic integer", that is*

$$x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n = 0 \qquad \textit{for some } a_1, \dots, a_n \in \mathbb{Z}$$

PROOF. Suppose that $f_1, \dots, f_n$ generate R as an Abelian group. Thus we must have the following for some integers

$$x \cdot f_1 = a_{11} f_1 + \dots + a_{1m} f_m$$
$$\vdots$$
$$x \cdot f_m = a_{m1} f_1 + \dots + a_{mm} f_m$$

Recall the Cayley-Hamilton theorem, the characteristic polynomial satisfies $\chi_A(A) = 0$. The matrix A in our case has integer entries so the characteristic equation has integer coefficients and either leading coefficients 1 (or -1 depending on how you define it &c.) Denote the polynomials

$$f(t) = t^n + a_1 t^{n-1} + \cdots + a_{n-1} t + a_n$$

Therefore $f(x) \cdot f_1 = 0, f(x) \cdot f_2 = 0, \ldots, f(x) \cdot f_m = 0$. Then the identity has some representation as a combination of the generators, and then if we multiply these by $f(x)$ they get annihilated and we obtain $f(x) \cdot 1 = f(x) = 0$ in R. □

$\frac{\# C_g \cdot \chi(g)}{\dim V}$ is the scalar by which $E_{\mathbf{C}_g}$ acts on V. $E_{\mathbf{C}_g}$, according to the lemma, satisfies a polynomial equation with integer coefficients in $\oplus_{\mathbf{C} \text{ conj. class}} \mathbb{Z} E_{\mathbf{C}}$. This polynomial equation remains valid in any representation. This finishes the proof of the first thing.

Now, in an attempt to bring this to some sort of conclusion we will discuss the following

23.1. Possible generalisations of the main results. We have always been dealing with finite groups and finite dimensional vector spaces and mainly also being only dealing with the case where the ground field was $\mathbb{C}$. So, there are three possible generalisations that we could consider. It is basically easy to deal with the case of infinite groups, in particular one type of group: the case of compact Lie groups. For example the case of $\mathbb{S}^1$, or the case $\mathbb{S}^3$ which has the structure commonly known as $\mathsf{SU}(2)$. In this case, the only major obstacle, is that we cannot do the "averaging" trick that we did. Instead, we replace this with an "averaging" process that is an integration, which can be made to make sense because of compactness.

Lecture 24

3/12 4pm

24.1. Possible generalisations of the main results: continued. In the case of compact Lie groups, if you were to apply the theory to $\mathbb{S}^1$, in particular the left regular representation, one would rediscover the theory of Fourier series. The theory involved in the case of $\mathsf{SU}(2), \mathsf{SO}(3)$ are involved in the case of quantum mechanical spin, and the structure of the periodic table. There is the notion of the "unitary trick" due to Hermann Weyl, which essentially amounts to understanding the representations of $\mathsf{GL}(\mathbb{C})$ which is done via restricting to the compact subgroup $\mathsf{U}_n(\mathbb{C})$ and dealing with representations in the universe of compact groups. The reason this works so well is because the representation theory of Lie groups is closely related to the representation theory of the associated Lie algebras - the linearity of the Lie algebras makes this a lot easier.

If you consider a finite group and take the field $\mathbb{C}$ but consider infinite dimensional vector spaces, then it is almost exactly the same - we can still use the "averaging operator". In particular

(1) every invariant subspace has an invariant complement.
(2) every representation has as invariant finite dimensional subspace

i.e. there is nothing really new here.

Finally, if we consider finite groups, and finite dimensional representations but vary the field, then there are two particular cases. The first is when the characteristic of the field divides the order of the group - in this case, it becomes very tricky, there are many open questions in this area. If the characteristic of the field does not divide the order then

(1) every invariant subspace has an invariant complement.
(2) the group algebra, rather the than being a direct sum of matrix algebras, is a direct sum of algebras over skew-fields (division algebras). For example

(a) Consider $G = \mathbb{Z}/4\mathbb{Z}$ and $\mathbb{F} = \mathbb{R}$. Over the complex numbers we would obtain four 1-dimensional representations corresponding to $\{1, -1, i, -i\}$ but over the real numbers we don't have the imaginary numbers, so these two become "glued" together into a 2-dimensional representation. We would have

$$\begin{array}{ll} \mathbb{R} & 1 \mapsto 1 \\ \mathbb{R} & 1 \mapsto -1 \\ \mathbb{R} & 1 \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{array}$$

and $\mathbb{R}G \simeq \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{C}$.

(b) In the case of D_4 the symmetries of the square, then all representations are defined over $\mathbb{R}$ and one can proove that

$$\mathbb{R}D_4 \simeq \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathrm{Mat}_2(\mathbb{R})$$

In the case of Q_8 the quaternions, all 1-dimensional representations are defined over real numbers but we also get a 4-dimensional irreducible real representation and (the same sort of "gluing" as described previously)

$$\mathbb{R}Q_8 \simeq \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{H}$$

Over $\mathbb{R}$, the only skew-fields (division algebras) are $\mathbb{R}, \mathbb{C}, \mathbb{H}$. Over the rational numbers there are infinitely many skew-fields.