MA 3416: Group representations
Selected answers/solutions to the assignment due February 2, 2015

1. A one-dimensional representation associates to each permutation σ a number $\rho(\sigma)$ such that $\rho(\sigma\tau) = \rho(\sigma)\rho(\tau)$. Note that (a) S_n is generated by transpositions so it is enough to know $\rho(\sigma)$ when σ is a transposition, (b) for a transposition, $\rho(\sigma)^2 = \rho(\sigma^2) = 1$, so each transposition is sent to ± 1, and hence every element is sent to ± 1, (c) $\rho((ij)(jk)) = \rho((ijk))$, and $\rho((ijk))^3 = 1$, so we note that a 3-cycle is, on the one hand, mapped to ± 1, and on the other hand, is mapped to a cubic root of 1. Hence all 3-cycles are mapped to 1, and all transpositions must be mapped to the same number. (Since $\rho((ij))\rho((jk)) = 1$, then two transpositions are mapped to the same number if they share an element, but then any two transpositions can be "connected" like that). If all transpositions are mapped to 1, we get the trivial representation, if all transpositions are mapped to -1, we get the sign representation.

2. (**a**) For the group D_4, check the general case of D_n below.

(**b**) It is easy to check that $[Q_8, Q_8] = \{1, -1\}$, and so there are 4 one-dimensional representations. Each of them sends both i and $-i$ to the same number equal to ± 1, both j and $-j$ to the same number equal to ± 1, and both k and $-k$ to the product of those.

3. Let us note that A_4 consists of the unit element, 3-cycles (each being an element of order 3), and products of two disjoint transpositions (each being an element of order 2). Under any one-dimensional representation, a 3-cycle must be represented by a cubic root of 1, and a product of two disjoint transpositions must be represented by ± 1. Note that in A_4 we have $(123)(234) = (12)(34)$, and so this (and any other) product of two disjoint transpositions is represented by a product of two cubic root of 1, so it must be represented by 1. We now note that $(123)(12)(34) = (134)$, $(123)(13)(24) = (243)$, and $(123)(14)(23) = (142)$. Therefore, (123), (134), (243), and (142) must be represented by the same cubic root of 1, and the remaining 4 3-cycles (the inverses of these) must be represented by the inverse of that cubic root. Altogether we get three one-dimensional representations.

4. One can take the counterclockwise rotation through $2\pi/n$ for a, and one of the reflections for b. The relations are clearly satisfied.

From the relation $ba = a^{-1}b$, it is easy to check that $[D_n, D_n]$ is generated by a^2. Since $a^n = 1$, we see that in the case of odd n, $[D_n, D_n]$ is equal to the subgroup generated by a, and so there are two one-dimensional representations (sending b to ± 1). In the case of even n, there are four representations, (sending each of a and b to ± 1).

5. Suppose that G is Abelian, that (V, ρ) is an irreducible representation. Take $g \in G$. Since for all $h \in G$ we have

$$\rho(g)\rho(h) = \rho(gh) = \rho(hg) = \rho(h)\rho(g),$$

we see that $\rho(g)$ is an intertwining operator, so is scalar by Schur's Lemma. Thus, all group elements act as scalars, and V has to be one-dimensional, otherwise any one-dimensional subspace is invariant.

6. It is clear that each of those linear operators has order 3, so it defines a representation. A two-dimensional representation is either irreducible or has an invariant one-dimensional subspace, which corresponds to a common eigenvector of all transformations representing group elements. But rotations of the real 2D plane have no eigenvectors.

7. We have $\rho_\pm(\bar{1}) = \begin{pmatrix} \cos\alpha & \mp\sin\alpha \\ \pm\sin\alpha & \cos\alpha \end{pmatrix}$, where $\alpha = 120^\circ$. Suppose that we have a homomorphism $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, so that $T\rho_+(\bar{1}) = \rho_-(\bar{1})T$. Writing these conditions in the matrix form, we conclude $a = -d$ and $b = c$, so every intertwining operator is of the form $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$; determinant of this matrix is $-a^2 - b^2$, so it is invertible unless $a = b = 0$, and so the representations are isomorphic.

8. Suppose that T is an intertwining operator from V to V, and that $T(e_j) = \sum_{i=1}^n a_{ij} e_i$. We have $\rho(\sigma)T = T\rho(\sigma)$, so

$$\sum_{i=1}^n a_{ij} e_{\sigma(i)} = Te_{\sigma(j)} = \sum_{k=1}^n a_{k\sigma(j)} e_k.$$

Denoting $k = \sigma(i)$ in the last sum, we have

$$\sum_{i=1}^n a_{ij} e_{\sigma(i)} = \sum_{i=1}^n a_{\sigma(i)\sigma(j)} e_{\sigma(i)},$$

so we must have $a_{ij} = a_{\sigma(i)\sigma(j)}$. Basically, this shows that all entries with $i = j$ must be equal to each other, and all entries with $i \neq j$ must be equal to each other, so the space of the intertwining operators is two-dimensional. This implies that this representation can be decomposed as a direct sum of two non-isomorphic irreducible representations. One of them is the trivial representation spanned by $e_1 + \cdots + e_n$, and the other one is its complement, which must have dimension $n - 1$.