MA 3416: Group representations
Selected answers/solutions to the assignment due April 2, 2015

1. First, we compute the values of ψ: $\psi(1^5) = \frac{2\cdots120}{24} = 10$ (as the sum goes over all $h \in S_5$), $\psi(21^3) = 0$ (as any conjugate of a transposition is again a transposition, and χ_U vanishes on transpositions), $\psi(31^2) = \frac{-1\cdot48}{24} = -2$ (as an element h contributes to the sum for, say, $g = (123)$ only if the image of $\{1,2,3\}$ under h is contained in $\{1,2,3,4\}$, so we should choose one element out of $\{1,2,3,4\}$ which is not in the image of h, a permutation of the rest of the three, and possibly swap the chosen element with 5, altogether $4 \times 6 \times 2 = 48$ choices), $\psi(2^21) = \frac{2\cdot24}{24} = 2$ (possible h are arbitrary elements of S_4), $\psi(41) = 0$ (same as for transpositions), $\psi(32) = \psi(5) = 0$, as no conjugates of these elements belong to S_4. Now it is easy to see that $\psi = \chi_W + \chi_{W'}$, so it is equal to the character of $W \oplus W'$.

2. (**a**) For such a group G, we would have $\#G = 1^2 + 1^2 + 1^2 + 1^2 + 5^2 = 29$. But 29 is a prime number, so the only group of order 29 is the cyclic group of that order, which is abelian, hence all its irreducible representations are one-dimensional. Therefore, there is no such group.

(**b**) There is always one conjugacy class consisting just of the unit element. Therefore, if we denote the the numbers of elements in the centralisers of the two other conjugacy classes n_1 and n_2, and the number of elements in G by n, we have $n = 1 + \frac{n}{n_1} + \frac{n}{n_2}$, or $1 = \frac{1}{n} + \frac{1}{n_1} + \frac{1}{n_2}$. Without loss of generality, $n > n_1 \geqslant n_2 \geqslant 2$ (the centraliser of an element g contains at least two elements, g and the unit). Note that $n_2 \leqslant 3$, for if $n_2 > 3$, we have $\frac{1}{n} + \frac{1}{n_1} + \frac{1}{n_2} < 3 \cdot \frac{1}{3} = 1$. (This also shows that if $n_2 = 3$, then $n = n_1 = n_2 = 3$, otherwise the sum of inverses is less than 1). If $n_2 = 2$, then $\frac{1}{2} = \frac{1}{n} + \frac{1}{n_1}$, and we have $n_1 \leqslant 4$, for if $n_1 > 4$, we have $\frac{1}{n} + \frac{1}{n_1} < 2 \cdot \frac{1}{4} = \frac{1}{2}$. (This also shows that if $n_1 = 4$, then $n = n_1 = 4$, otherwise the sum of inverses is less than $\frac{1}{2}$). If $n_1 = 2$, then we have $0 = \frac{1}{n}$, a contradiction. If $n_1 = 3$, then we have $n = 6$. Altogether, we see that there are at most three choices for (n, n_1, n_2) for which a group with three conjugacy classes may exist: $(3,3,3)$, $(4,4,2)$, and $(6,3,2)$. In the first case, $\#G = 3$, so G is Abelian cyclic, and indeed has three conjugacy classes. In the second case, $\#G = 4$, so G is Abelian, and therefore has four conjugacy classes. In the last case, $\#G = 6$, so G is either Abelian cyclic, and then it has six conjugacy classes, or S_3, and then it has three conjugacy classes.

(**c**) The number of complex irreducible representations of a group is equal to the number of conjugacy classes, so such group would have three conjugacy classes. All such groups are classified in the previous question, and they are of order at most 6.

3. (**a**) Let us consider the linear map $T_k\colon U_k \to U_{n-k}$ that sends every basis element e_A corresponding to a subset A to the element $e_{\{1,\ldots,n\}\setminus A}$ corresponding to the complement of A. This map is manifestly a vector space isomorphism, since we have $T_{n-k}T_k = \mathrm{Id}_{U_k}$ and $T_kT_{n-k} = \mathrm{Id}_{U_{n-k}}$. It is also an intertwining operator, since computing complements commutes with permutations. Therefore, these representations are isomorphic.

(**b**) The previous question shows that it is enough to consider the case $k, l \leqslant n/2$, for otherwise we can replace the representations by isomorphic ones so that these inequalities hold. We know that the dimension of the space of intertwining operators is equal to the number of orbits on the product $P_k \times P_l$, where P_i is the set of all i-element subsets. An orbit of the pair $(A, B) \in P_k \times P_l$ is completely determined by the size of the intersection $A \cap B$, which can be any integer nor exceeding $\min(k, l)$, so the number of orbits is $1 + \min(k, l)$.

Recalling that we may have needed to adjust k and l to fulfil $k, l \leqslant n/2$, we write the answer as $1 + \min(k, l, n-k, n-l)$.

4. As before, it is enough to show that for $k \leqslant n/2$. Let us prove that there exist irreducible representations $V_0, V_1, \dots, V_{\lfloor n/2 \rfloor}$ such that for each $k \leqslant n/2$ we have

$$U_k \simeq V_0 \oplus V_1 \oplus \cdots \oplus V_k.$$

We shall prove the existence of V_k by induction on k. For $k = 0$, the statement is obvious, since U_0 is isomorphic to the trivial irreducible representation. Assume that we know it for some k, and would like to prove it for $k+1 \leqslant n/2$. By previous question, we have $\dim \mathsf{Hom}_{S_n}(U_i, U_{k+1}) = \dim \mathsf{Hom}_{S_n}(V_0 \oplus \cdots \oplus V_i, U_{k+1}) = i + 1$ for $i \leqslant k$, therefore we see that the multiplicity of V_0 in U_{k+1} is one, the sum of the multiplicity of V_0 and the multiplicity of V_1 in U_{k+1} is two, etc., hence each of $V_0, \dots, V_k$ has the multiplicity one in U_{k+1}. Note also that $\dim \mathsf{Hom}_{S_n}(U_{k+1}, U_{k+1}) = k+2$, so U_{k+1} has precisely one more irreducible constituent V_{k+1}, and it is different from all the V_i for $i \leqslant k$, otherwise the corresponding dimension would have been $2^2 + k = k + 4 \neq k + 2$.

Finally, the ring of intertwining operators on U_k is isomorphic to a product of $k+2$ copies of $\mathbb{C}$, because on each irreducible constituent an intertwining operator is a scalar. Therefore, the ring of intertwining operators is commutative.