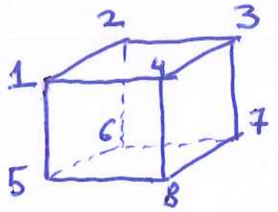
MA 3416: Group representations
Homework problems due March 4, 2015

In all questions below the ground field is $\mathbb{C}$.

**1.** Find multiplicities of irreducibles in all pairwise tensor products of irreducible representations of $S_4$.

**2.** **(a)** Consider the action of $S_4$ (viewed as the group of rotations of the cube) on the space $V$ of all functions on vertices of the cube. Compute the character of this representations, find multiplicities of irreducibles in it, and describe the invariant subspaces where those irreducibles are realised.

**(b)** Let $T\colon V \to V$ be a linear transformation of the space $V$ of all functions on vertices of the cube that maps a function $f$ to the function $Tf$ whose value on each vertex $v$ is the average of values of $f$ on the three neighbours of $v$. Compute the limit $\lim\limits_{n\to\infty} T^n(w)$, where $w$ is the following function:




Recall that the $n$th exterior power of a vector space $W$ (which is denoted by $\Lambda^n(W)$) is a subspace in its $n$th tensor power $W^{\otimes n}$ which is spanned by all skew-symmetric products

$$w_1 \wedge w_2 \wedge \ldots \wedge w_n = \frac{1}{n!} \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) w_{\sigma(1)} \otimes w_{\sigma(2)} \otimes \ldots \otimes w_{\sigma(n)}$$

for all $w_1, \ldots, w_n \in W$. The $n$th exterior power $\Lambda^n(A)$ of an operator $A\colon W \to W$ is defined by

$$\Lambda^n(A)(w_1 \wedge w_2 \wedge \ldots \wedge w_n) = (Aw_1) \wedge (Aw_2) \wedge \ldots \wedge (Aw_n).$$

If $(W, \rho)$ is a representation of a finite group $G$, $(\Lambda^n(W), \Lambda^n(\rho))$ is an invariant subspace of $(W^{\otimes n}, \rho^{\otimes n})$ which is called the $n$th exterior power of the representation $W$.

**3.** Express traces of exterior powers of a linear transformation $A$ via the coefficients of the characteristic polynomial of $A$. (For simplicity, assume that $A$ can be diagonalised).

**4.** Prove that $\chi_{\Lambda^2(V)}(g) = \frac{1}{2}(\chi_V(g)^2 - \chi_V(g^2))$.

**5.** Compute multiplicities of irreducibles in the following representations of $S_4$: **(a)** $\Lambda^2(V)$, $\Lambda^2(V')$, and $\Lambda^2(U)$; **(b)** $\Lambda^3(V)$, $\Lambda^3(V')$, and $\Lambda^3(U)$.