MA 3416: Group representations
Homework problems due March 23, 2015

1. Let (V, ρ) be a complex representation of a finite group G.

(**a**) Prove that $G_\rho = \{g \in G \mid \rho(g) = \lambda \mathrm{Id}_V \text{ for some } \lambda \in \mathbb{C}\}$ is a normal subgroup of G.

(**b**) Prove that for any $g \in G$ we have $|\chi_V(g)| \leqslant \dim(V)$. (*Hint*: trace of an operator is equal to the sum of its eigenvalues.)

(**c**) Prove that $|\chi_V(g)| = \dim(V)$ if and only if g belongs to the subgroup G_ρ from the previous problem.

2. Consider the action of S_4 on each of the following sets:

(**a**) set of vertices of the tetrahedron;

(**b**) set of edges of the tetrahedron;

(**c**) set of faces of the tetrahedron;

(**d**) set of edges of the cube.

Each of these actions leads to a representation of S_4 on the vector space of functions on the corresponding finite set. Compute the characters of these representations, find multiplicities of irreducibles in them, and describe the corresponding invariant subspaces where those irreducibles are realised.

3. Find multiplicities of irreducible representations of A_5 in $\mathbb{C}M$, where M is the set of faces of the dodecahedron.

4. For each two of five complex irreducible representations of A_5, find multiplicities of irreducibles in their tensor product.

5. Suppose that G_1 and G_2 are two finite groups.

(**a**) Describe the conjugacy classes of $G_1 \times G_2$, assuming conjugacy classes of G_1 and G_2 known.

(**b**) Show that if (V_1, ρ_1) is a complex irreducible representation of G_1 and (V_2, ρ_2) is a complex irreducible representation of G_2, then $(V_1 \otimes V_2, \rho_1 \boxtimes \rho_2)$, where

$$(\rho_1 \boxtimes \rho_2)(g_1, g_2) = \rho_1(g_1) \otimes \rho_2(g_2),$$

is a complex irreducible representation of $G_1 \times G_2$. Show that every complex irreducible representation of $G_1 \times G_2$ can be obtained this way.