MA 3419: Galois theory
Homework problems due October 31, 2017

Solutions to this are due by the end of the 11am class on Tuesday October 31. Please attach a cover sheet with a declaration `http://tcd-ie.libguides.com/plagiarism/declaration` confirming that you know and understand College rules on plagiarism.

Please put your name and student number on each of the sheets you are handing in. (Or, ideally, staple them together).

1. Let $\alpha \in \mathbb{C}$ be one of the roots of $x^3 - x - 1$, and $\beta \in \mathbb{C}$ be one of the roots of $x^3 - x - \alpha$. Write some polynomial with rational coefficients that has β as a root.

2. Let $k \subset K$ be a field extension, and let $\alpha, \beta \in K$. Suppose that $[k(\alpha)\colon k] = m$ and $[k(\beta)\colon k] = n$. Show that $[k(\alpha, \beta)\colon k(\alpha)] = n$ if and only if $[k(\alpha, \beta)\colon k(\beta)] = m$. Does either of these equivalent conditions hold for $\alpha = \sqrt[3]{2}$ and $\beta = \omega\sqrt[3]{2}$, where ω is the primitive complex cube root of 1?

3. Compute the degree of the splitting field of $x^4 - 2$ over $\mathbb{Q}$, and find some basis for that extension as a $\mathbb{Q}$-vector space.

4. Determine the Galois group of $x^4 - 2$ over $\mathbb{Q}$.

5. Let p be a prime number, let k be a field (no assumption on characteristic), and let $k \subset K$ be a field extension. Show that if p is coprime to $[K\colon k]$, then $a \in k$ is a p-th power in k if and only if it is a p-th power in K.

6. Is the extension $\mathbb{F}_2 \subset \mathbb{F}_8$ a Galois extension? Why? Find the Galois group of this extension. (*Hint*: over a field of characteristic two, the map $x \mapsto x^2$ is an automorphism).