Assignment 3

MA341C — Seminar on *Proofs from THE BOOK*

Trinity College Dublin

NAME AND SURNAME: ....

STUDENT NUMBER: .... NUMBER OF PAGES: ....

Note: solutions to this assignment are due by 11am on Wednesday, October 3rd. Please attach a cover sheet with a declaration (http://tcd-ie.libguides.com/plagiarism/declaration) confirming that you know and understand College rules on plagiarism. All exercises are weighed equally unless otherwise stated.

Exercise 1. Consider the number

$$x = \sum_{n=0}^{\infty} \frac{1}{10^{n!}}.$$

(i) Show that x is irrational.
(ii) Show that x^2 is irrational.

Exercise 2. Suppose that $\cos\alpha = \frac{3}{5}$. Show that $\frac{\alpha}{\pi}$ is irrational. (*Hint:* you might want to use that the ring $\mathbb{Z}[i]$ of Gaussian integers is a UFD.)

In the next two questions, $G = (V, E)$ is a finite simple graph.

Exercise 3. The independence number $\alpha(G)$ of G is the maximal number of pairwise nonadjacent vertices in G. Prove the dual version of Turán's Theorem: if G has n vertices and $\frac{nk}{2}$ edges, for $k \geq 1$, then $\alpha(G) \geq \frac{n}{(k+1)}$.

Exercise 4. Denote by $t(G)$ the number of triangles in G. If G has n vertices and m edges, show that

$$t(G) + t(G^c) \geq \binom{n}{3} + \frac{2m^2}{n} - m(n-1),$$

where G^c is the complement graph. (*Hint:* Let t_i, for each vertex i of G, be the number of ways to choose two more vertices $\{j, k\}$ so that the vertex i is adjacent to precisely one of them. Find a relationship between $t(G) + t(G^c)$ and $\sum_i t_i$, and express t_i via the degree of the vertex i.)