SOME IRRATIONAL NUMBERS

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1. Introduction

The first known proof of a number being irrational is older than Euclid himself; a Pythagorean, assuming the square root of two was rational, reached a contradiction, showing it to be in fact rational. Stunning as it was at the time (and allegedly fatal for the discoverer), it was to be another fifteen centuries or so until a proof was found showing any other numbers, excepting the square root of a square-free integer, of being irrational. In this paper, we give three such proofs:

- e is irrational
- e^s is irrational for $s \in \mathbb{Q} \setminus \{0\}$
- π^2 is irrational.

In proving the last theorem we then obtain as an easy corollary that π is irrational.

2. e is irrational

The following theorem is due to Fourier.

Theorem. *e is irrational*

Proof. Assume $e = \sum_{k=0}^{\infty} 1/k! = a/b$, the ratio of positive integers. We then have $n!be = n!a$ for any integer n. The right hand side is an integer, while expanding $n!be$ gives

$$n!be = n!b(\sum_{k=0}^{\infty} \frac{1}{k!}) = n!b(1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!}) + n!b(\frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \dots).$$

The first term in this sum is an integer, as all factorials less than n divide $n!$; for the second term we have

$$\frac{b}{n+1} < n!b(\frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \dots) < b(\frac{1}{(n+1)} + \frac{1}{(n+1)^2} + \dots) = \frac{b}{n}.$$

implying for large enough n (take $n = 2b$ for example) we have $0 < n!b(\frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \dots) < 1$, showing $n!be$ to not be an integer, a contradiction. □

3. e^s is irrational for $r \in \mathbb{Q} \setminus \{0\}$

We first prove the following lemma.

Lemma. *Define the function $f(x) = \frac{x^n(1-x)^n}{n!}$, then*

(1) f is a polynomial of the form $\frac{1}{n!}\sum_{k=n}^{2n} c_k x^k$, $c_k \in \mathbb{Z}$
(2) for $0 < x < 1$, $0 < f(x) < 1/n!$
(3) for $k \in \mathbb{N}$ we have $f^{(k)}(0), f^{(k)}(1) \in \mathbb{Z}$.

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Proof. (1) & (2) are obvious. For (3) we have $f^{(k)}(0) = 0$ when $0 \leq k < n$; while $f^{(k)}(0) = \frac{c_k}{n!}k! \in \mathbb{Z}$ for $n \leq k \leq 2n$. Noting that $f(x) = f(1-x)$ we get $f^{(k)}(x) = (-1)^k f^{(k)}(1-x)$ by the chain rule, implying $f(1) = (-1)^k f^k(0) \in \mathbb{Z}$, giving the result. □

Now for the proof; we consider it in two cases.

Theorem. *e^s is irrational for $s \in \mathbb{Q} \setminus \{0\}$*

Proof. Assume $e^s = a/b$, the ratio of positive integers; for the first case we assume s is a positive integer. Choose n such that $n! > as^{2n+1}$, for reasons that will become clear shortly. Put

$$F(x) = s^2nf(x) - s^{2n-1}f'(x) + s^{2n-2}f''(x) - \cdots + f^{(2n)}(x) - f^{(2n+1)}(x) + \ldots$$

The higher derivatives greater than $2n$ vanish, but writing in this way gives the identity $F'(x) = -sF(x) + s^{2n+1}f(x)$, which implies $\frac{d}{dx}(e^{sx}F(x)) = e^{sx}s^{2n+1}f(x)$. Now, for a contradiction, put

$$N = b\int_0^1 e^{2n+1}e^{sx}f(x)dx = b[e^{sx}F(x)]_0^1 = be^sF(1) - bF(0) = aF(1) - bF(0),$$

which is an integer by the previous lemma, but then

$$N = b\int_0^1 e^{2n+1}e^{sx}f(x)dx < \frac{bs^{2n+1}e^s}{n!} = \frac{as^{2n+1}}{n!} < 1,$$

also by the previous lemma, a contradiction.

For the second case, we assume $s \in \mathbb{Q}\setminus\{0\}$. If $e^s = e^{\frac{a}{b}}$ is rational then $(e^{\frac{a}{b}})^b = e^a$ would be rational, in contradiction to the first case. □

4. π and π^2 are irrational

We re-use the polynomial f defined above for the following. We explicitly assume π is positive, which is clear from the identity $\pi = \int_{-1}^1 \frac{1}{\sqrt{1-x^2}}dx$.

Theorem. *π^2 is irrational*

Proof. Assume $\pi = a/b$, the ratio of positive integers. Putting

$$F(x) = b^n(\pi^{2n}f(x) - \pi^{2n-2}f'(x) + \pi^{2n-4}f''(x) - \ldots),$$

we then have

$$F''(x) = -\pi^2F(x) + b^n\pi^{2n+2}f(x),$$

implying

$$\frac{d}{dx}(F'(x)\sin\pi x - \pi F(x)\cos\pi x) = \pi^2a^nf(x)\sin\pi x.$$

Define

$$N = \pi\int_0^1 a^nf(x)\sin\pi x dx = [\frac{1}{\pi}f'(x)\sin\pi x - F(x)\cos\pi x]_0^1 = F(0) + F(1),$$

which is again an integer from the previous lemma.

Choose n such that $\pi a^n < n!$ then

$$0 < N = \pi\int_0^1 a^nf(x)\sin\pi x dx < \frac{\pi a^n}{n!} < 1,$$

a contradiction. □

Corollary. π *is irrational*

Proof. If $\pi = \frac{a}{b}$ was the ratio of positive integers, then $\pi^2 = \frac{a^2}{b^2}$ would be rational, in contradiction to the previous theorem. ☐

References

[1] Martin Aigner and Gunter M. Ziegler. *Proofs from The Book.* Springer-Verlag, Berlin, fifth edition, 2014. Including illustrations by Karl H. Hofmann.